Problem 4.1 :

(a)

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(a)}{t-a}da$$

Hence :

$$\begin{array}{rcl} -\hat{x}(-t) & = & -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(a)}{-t-a}da \\ & = & -\frac{1}{\pi}\int_{\infty}^{-\infty}\frac{x(-b)}{-t+b}(-db) \\ & = & -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(b)}{-t+b}db \\ & = & \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(b)}{t-b}db = \hat{x}(t) \end{array}$$

where we have made the change of variables : $b = -a$ and used the relationship : $x(b) = x(-b)$.

(b) In exactly the same way as in part (a) we prove :

$$\hat{x}(t) = \hat{x}(-t)$$

(c) $x(t) = \cos\omega_0 t$, so its Fourier transform is : $X(f) = \frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right],\ f_0 = 2\pi\omega_0$. Exploiting the phase-shifting property (4-1-7) of the Hilbert transform :

$$\hat{X}(f) = \frac{1}{2}\left[-j\delta(f-f_0)+j\delta(f+f_0)\right] = \frac{1}{2j}\left[\delta(f-f_0)-\delta(f+f_0)\right] = F^{-1}\left\{\sin 2\pi f_0 t\right\}$$

Hence, $\hat{x}(t) = \sin\omega_0 t$.

(d) In a similar way to part (c) :

$$x(t) = \sin\omega_0 t \Rightarrow X(f) = \frac{1}{2j}\left[\delta(f-f_0)-\delta(f+f_0)\right] \Rightarrow \hat{X}(f) = \frac{1}{2}\left[-\delta(f-f_0)-\delta(f+f_0)\right]$$

$$\Rightarrow \hat{X}(f) = -\frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right] = -F^{-1}\left\{\cos 2\pi\omega_0 t\right\} \Rightarrow \hat{x}(t) = -\cos\omega_0 t$$

(e) The positive frequency content of the new signal will be : $(-j)(-j)X(f) = -X(f),\ f>0$, while the negative frequency content will be : $j\cdot jX(f) = -X(f),\ f<0$. Hence, since $\hat{\hat{X}}(f) = -X(f)$, we have : $\hat{\hat{x}}(t) = -x(t)$.

(f) Since the magnitude response of the Hilbert transformer is characterized by : $|H(f)| = 1$, we have that : $\left|\hat{X}(f)\right| = |H(f)|\,|X(f)| = |X(f)|$. Hence :

$$\int_{-\infty}^{\infty} \left|\hat{X}(f)\right|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and using Parseval's relationship :

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$$

(g) From parts (a) and (b) above, we note that if $x(t)$ is even, $\hat{x}(t)$ is odd and vice-versa. Therefore, $x(t)\hat{x}(t)$ is always odd and hence : $\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = 0$.

Problem 4.2 :

We have :

$$\hat{x}(t) = h(t) * x(t)$$

where $h(t) = \frac{1}{\pi t}$ and $H(f) = \left\{ \begin{array}{cc} -j, & \text{f} > 0 \\ j, & \text{f} < 0 \end{array} \right\}$. Hence :

$$\Phi_{\hat{x}\hat{x}}(f) = \Phi_{xx}(f)\,|H(f)|^2 = \Phi_{xx}(f)$$

and its inverse Fourier transform :

$$\phi_{\hat{x}\hat{x}}(\tau) = \phi_{xx}(\tau)$$

Also :

$$\begin{aligned} \phi_{x\hat{x}}(\tau) &= E\left[x(t+\tau)\hat{x}(t)\right] \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{E[x(t+\tau)x(a)]}{t-a} da \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi_{xx}(t+\tau-a)}{t-a} da \\ &= -\frac{1}{\pi}\int_{\infty}^{-\infty} \frac{\phi_{xx}(b)}{b-\tau} db \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\phi_{xx}(b)}{\tau-b} db = -\hat{\phi}_{xx}(\tau) \end{aligned}$$

Problem 4.3 :

(a)

$$\begin{aligned} E\left[z(t)z(t+\tau)\right] &= E\left[\{x(t+\tau)+jy(t+t)\}\{x(t)+jy(t)\}\right] \\ &= E\left[x(t)x(t+\tau)\right]-E\left[y(t)y(t+\tau)\right]+jE\left[x(t)y(t+\tau)\right] \\ &\quad +E\left[y(t)x(t+\tau)\right] \\ &= \phi_{xx}(\tau)-\phi_{yy}(\tau)+j\left[\phi_{yx}(\tau)+\phi_{xy}(\tau)\right] \end{aligned}$$

But $\phi_{xx}(\tau)=\phi_{yy}(\tau)$and $\phi_{yx}(\tau)=-\phi_{xy}(\tau)$. Therefore :

$$E\left[z(t)z(t+\tau)\right]=0$$

(b)

$$V=\int_0^T z(t)dt$$

$$E\left(V^2\right)=\int_0^T\int_0^T E\left[z(a)z(b)\right]dadb=0$$

from the result in (a) above. Also :

$$\begin{aligned} E\left(VV^*\right) &= \int_0^T\int_0^T E\left[z(a)z^*(b)\right]dadb \\ &= \int_0^T\int_0^T 2N_0\delta(a-b)dadb \\ &= \int_0^T 2N_0 da = 2N_0T \end{aligned}$$

Problem 4.9 :

The energy of the signal waveform $s'_m(t)$ is :

$$\begin{aligned}
\mathcal{E}' &= \int_{-\infty}^{\infty} |s'_m(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} s_m^2(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt \\
&\quad -\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt - \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt \\
&= \mathcal{E} + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\mathcal{E}\delta_{kl} - \frac{2}{M}\mathcal{E} \\
&= \mathcal{E} + \frac{1}{M}\mathcal{E} - \frac{2}{M}\mathcal{E} = \left(\frac{M-1}{M}\right)\mathcal{E}
\end{aligned}$$

The correlation coefficient is given by :

$$\begin{aligned}
\rho_{mn} &= \frac{1}{\mathcal{E}'}\int_{-\infty}^{\infty} s'_m(t)s'_n(t)dt = \frac{1}{\mathcal{E}'}\int_{-\infty}^{\infty}\left(s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t)\right)\left(s_n(t) - \frac{1}{M}\sum_{l=1}^{M} s_l(t)\right) dt \\
&= \frac{1}{\mathcal{E}'}\left(\int_{-\infty}^{\infty} s_m(t)s_n(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt\right) \\
&\quad -\frac{1}{\mathcal{E}'}\left(\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_n(t)s_k(t)dt + \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt\right) \\
&= \frac{\frac{1}{M^2}M\mathcal{E} - \frac{1}{M}\mathcal{E} - \frac{1}{M}\mathcal{E}}{\frac{M-1}{M}\mathcal{E}} = -\frac{1}{M-1}
\end{aligned}$$

Problem 4.17 :

The first basis function is :

$$g_4(t) = \frac{s_4(t)}{\sqrt{\mathcal{E}_4}} = \frac{s_4(t)}{\sqrt{3}} = \left\{ \begin{array}{cc} -1/\sqrt{3}, & 0 \leq t \leq 3 \\ 0, & \text{o.w.} \end{array} \right\}$$

Then, for the second basis function :

$$c_{43} = \int_{-\infty}^{\infty} s_3(t) g_4(t) dt = -1/\sqrt{3} \Rightarrow g_3'(t) = s_3(t) - c_{43} g_4(t) = \left\{ \begin{array}{cc} 2/3, & 0 \leq t \leq 2 \\ -4/3, & 2 \leq t \leq 3 \\ 0, & \text{o.w} \end{array} \right\}$$

Hence :

$$g_3(t) = \frac{g_3'(t)}{\sqrt{E_3}} = \left\{ \begin{array}{cc} 1/\sqrt{6}, & 0 \leq t \leq 2 \\ -2/\sqrt{6}, & 2 \leq t \leq 3 \\ 0, & \text{o.w} \end{array} \right\}$$

where E_3 denotes the energy of $g_3'(t)$: $E_3 = \int_0^3 (g_3'(t))^2 \, dt = 8/3$.
For the third basis function :

$$c_{42} = \int_{-\infty}^{\infty} s_2(t) g_4(t) dt = 0 \quad \text{and} \quad c_{32} = \int_{-\infty}^{\infty} s_2(t) g_3(t) dt = 0$$

Hence :

$$g_2'(t) = s_2(t) - c_{42} g_4(t) - c_{32} g_3(t) = s_2(t)$$

and

$$g_2(t) = \frac{g_2'(t)}{\sqrt{\mathcal{E}_2}} = \left\{ \begin{array}{cc} 1/\sqrt{2}, & 0 \leq t \leq 1 \\ -1/\sqrt{2}, & 1 \leq t \leq 2 \\ 0, & \text{o.w} \end{array} \right\}$$

where : $\mathcal{E}_2 = \int_0^2 (s_2(t))^2 \, dt = 2$.
Finally for the fourth basis function :

$$c_{41} = \int_{-\infty}^{\infty} s_1(t) g_4(t) dt = -2/\sqrt{3}, \; c_{31} = \int_{-\infty}^{\infty} s_1(t) g_3(t) dt = 2/\sqrt{6}, \; c_{21} = 0$$

Hence :

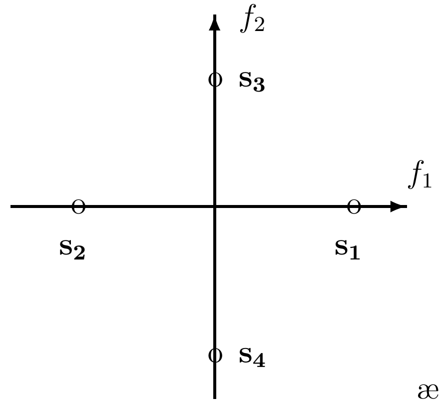
$$g_1'(t) = s_1(t) - c_{41} g_4(t) - c_{31} g_3(t) - c_{21} g_2(t) = 0 \Rightarrow g_1(t) = 0$$

The last result is expected, since the dimensionality of the vector space generated by these signals is 3. Based on the basis functions $(g_2(t), g_3(t), g_4(t))$ the basis representation of the signals is :

$$
\begin{aligned}
\mathbf{s}_4 &= \left(0, 0, \sqrt{3}\right) \Rightarrow \mathcal{E}_4 = 3 \\
\mathbf{s}_3 &= \left(0, \sqrt{8/3}, -1/\sqrt{3}\right) \Rightarrow \mathcal{E}_3 = 3 \\
\mathbf{s}_2 &= \left(\sqrt{2}, 0, 0\right) \Rightarrow \mathcal{E}_2 = 2 \\
\mathbf{s}_1 &= \left(2/\sqrt{6}, -2/\sqrt{3}, 0\right) \Rightarrow \mathcal{E}_1 = 2
\end{aligned}
$$

Problem 4.18 :

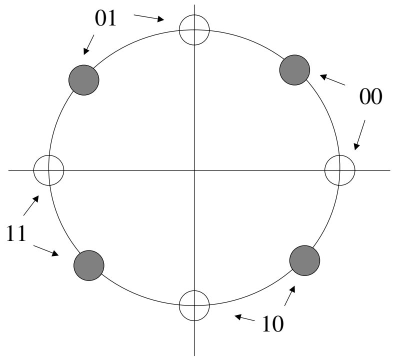
$$\begin{aligned} \mathbf{s}_1 &= \left(\sqrt{\mathcal{E}}, 0\right) \\ \mathbf{s}_2 &= \left(-\sqrt{\mathcal{E}}, 0\right) \\ \mathbf{s}_3 &= \left(0, \sqrt{\mathcal{E}}\right) \\ \mathbf{s}_4 &= \left(0, -\sqrt{\mathcal{E}}\right) \end{aligned}$$



As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

Problem 4.19 :

(a)(b) The signal space diagram, together with the Gray encoding of each signal point is given in the following figure :



The signal points that may be transmitted at times $t = 2nT \;\; n = 0, 1, ...$ are given with blank circles, while the ones that may be transmitted at times $t = 2nT + 1, \;\; n = 0, 1, ...$ are given with filled circles.