Problem 4.15:

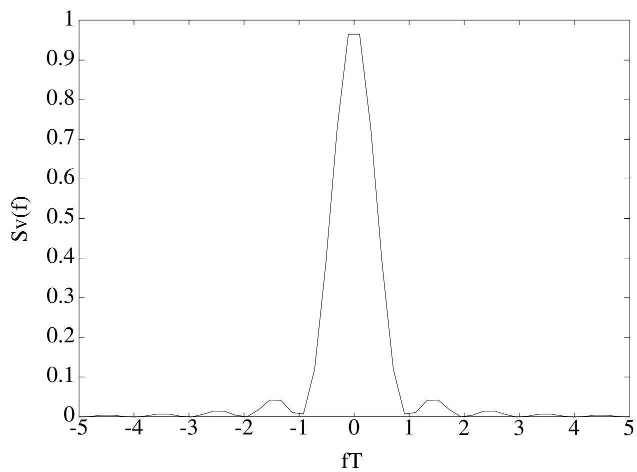
We have that $\Phi_{uu}(f) = \frac{1}{T}\left|G(f)\right|^2 \Phi_{ii}(f)$ But $\mathrm{E}(I_n) = 0,\ E\left(\left|I_n\right|^2\right) = 1$, hence : $\phi_{ii}(m) = \left\{ \begin{array}{ll} 1, & m=0 \\ 0, & m \neq 0 \end{array} \right\}$. Therefore : $\Phi_{ii}(f) = 1 \Rightarrow \Phi_{uu}(f) = \frac{1}{T}\left|G(f)\right|^2$.

(a) For the rectangular pulse :

$$G(f) = AT\frac{\sin \pi f T}{\pi f T}e^{-j2\pi f T/2} \Rightarrow \left|G(f)\right|^2 = A^2T^2\frac{\sin^2 \pi f T}{(\pi f T)^2}$$

where the factor $e^{-j2\pi f T/2}$ is due to the $T/2$ shift of the rectangular pulse from the center $t=0$. Hence :

$$\Phi_{uu}(f) = A^2T\frac{\sin^2 \pi f T}{(\pi f T)^2}$$

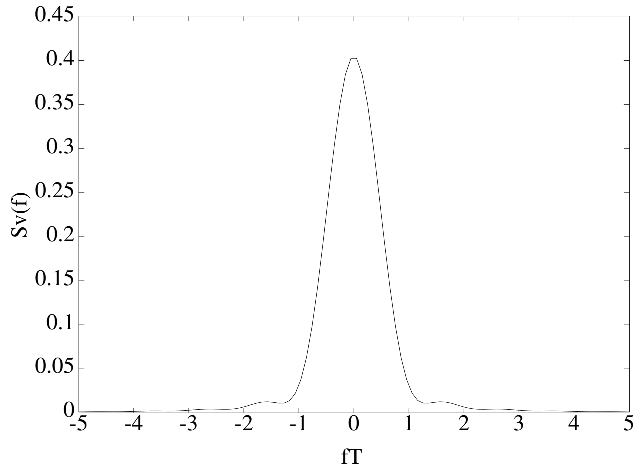


(b) For the sinusoidal pulse : $G(f) = \int_0^T \sin\frac{\pi t}{T}\exp(-j2\pi f t)dt$. By using the trigonometric identity $\sin x = \frac{\exp(jx)-\exp(-jx)}{2j}$ it is easily shown that :

$$G(f) = \frac{2AT}{\pi}\frac{\cos \pi T f}{1-4T^2f^2}e^{-j2\pi f T/2} \Rightarrow \left|G(f)\right|^2 = \left(\frac{2AT}{\pi}\right)^2\frac{\cos^2 \pi T f}{(1-4T^2f^2)^2}$$

Hence :

$$\Phi_{uu}(f) = \left(\frac{2A}{\pi}\right)^2 T\frac{\cos^2 \pi T f}{(1-4T^2f^2)^2}$$



(c) The 3-db frequency for (a) is :

$$\frac{\sin^2 \pi f_{3db} T}{(\pi f_{3db} T)^2} = \frac{1}{2} \Rightarrow f_{3db} = \frac{0.44}{T}$$

(where this solution is obtained graphically), while the 3-db frequency for the sinusoidal pulse on (b) is :

$$\frac{\cos^2 \pi T f}{(1 - 4T^2 f^2)^2} = \frac{1}{2} \Rightarrow f_{3db} = \frac{0.59}{T}$$

The rectangular pulse spectrum has the first spectral null at $f = 1/T$, whereas the spectrum of the sinusoidal pulse has the first null at $f = 3/2T = 1.5/T$. Clearly the spectrum for the rectangular pulse has a narrower main lobe. However, it has higher sidelobes.

Problem 4.20 :

The autocorrelation function for $u_\Delta(t)$ is :

$$\begin{aligned}
\phi_{u_\Delta u_\Delta}(t) &= \tfrac{1}{2}E\left[u_\Delta(t+\tau)u_\Delta^*(t)\right] \\
&= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left(I_m I_n^*\right) E\left[u(t+\tau-mT-\Delta)u^*(t-nT-\Delta)\right] \\
&= \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \phi_{ii}(m-n) E\left[u(t+\tau-mT-\Delta)u^*(t-nT-\Delta)\right] \\
&= \tfrac{1}{2}\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\sum_{n=-\infty}^{\infty} E\left[u(t+\tau-mT-nT-\Delta)u^*(t-nT-\Delta)\right] \\
&= \tfrac{1}{2}\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\sum_{n=-\infty}^{\infty}\int_0^T \tfrac{1}{T}u(t+\tau-mT-nT-\Delta)u^*(t-nT-\Delta)d\Delta
\end{aligned}$$

Let $a=\Delta+nT,\ da=d\Delta,$ and $a\in(-\infty,\infty)$. Then :

$$\begin{aligned}
\phi_{u_\Delta u_\Delta}(t) &= \tfrac{1}{2}\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\sum_{n=-\infty}^{\infty}\int_{nT}^{(n+1)T}\tfrac{1}{T}u(t+\tau-mT-a)u^*(t-a)da \\
&= \tfrac{1}{2}\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\tfrac{1}{T}\int_{-\infty}^{\infty}u(t+\tau-mT-a)u^*(t-a)da \\
&= \tfrac{1}{T}\sum_{m=-\infty}^{\infty}\phi_{ii}(m)\phi_{uu}(\tau-mT)
\end{aligned}$$

Thus we have obtained the same autocorrelation function as given by (4.4.11). Consequently the power spectral density of $u_\Delta(t)$ is the same as the one given by (4.4.12) :

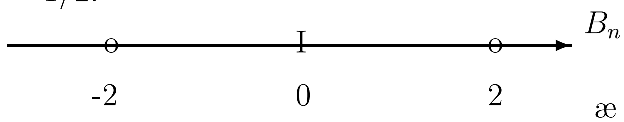
$$\Phi_{u_\Delta u_\Delta}(f)=\frac{1}{T}\left|G(f)\right|^2\Phi_{ii}(f)$$

Problem 4.21 :

(a) $B_n = I_n + I_{n-1}$. Hence :

I_n	I_{n-1}	B_n
1	1	2
1	−1	0
−1	1	0
−1	−1	−2

The signal space representation is given in the following figure, with $P(B_n = 2) = P(B_n = -2) = 1/4, \quad P(B_n = 0) = 1/2$.



(b)

$$\begin{aligned} \phi_{BB}(m) &= E\left[B_{n+m}B_n\right] = E\left[\left(I_{n+m} + I_{n+m-1}\right)\left(I_n + I_{n-1}\right)\right] \\ &= \phi_{ii}(m) + \phi_{ii}(m-1) + \phi_{ii}(m+1) \end{aligned}$$

Since the sequence $\{I_n\}$ consists of independent symbols :

$$\phi_{ii}(m) = \left\{ \begin{array}{cc} E\left[I_{n+m}\right]E\left[I_n\right] = 0 \cdot 0 = 0, & \text{m} \neq 0 \\ E\left[I_n^2\right] = 1, & \text{m} = 0 \end{array} \right\}$$

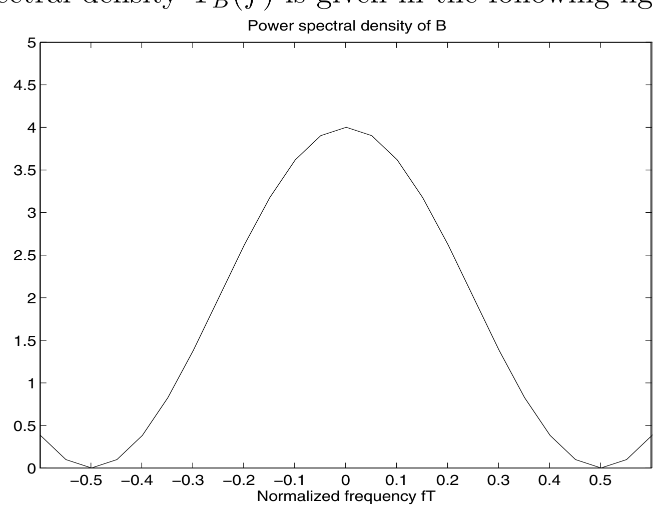
Hence :

$$\phi_{BB}(m) = \left\{ \begin{array}{cc} 2, & \text{m} = 0 \\ 1, & \text{m} = \pm 1 \\ 0, & \text{o.w} \end{array} \right\}$$

and

$$\begin{aligned} \Phi_{BB}(f) &= \textstyle\sum_{m=-\infty}^{\infty} \phi_{BB}(m) \exp(-j2\pi f m T) = 2 + \exp(j2\pi f T) + \exp(-j2\pi f T) \\ &= 2\left[1 + \cos 2\pi f T\right] = 4\cos^2 \pi f T \end{aligned}$$

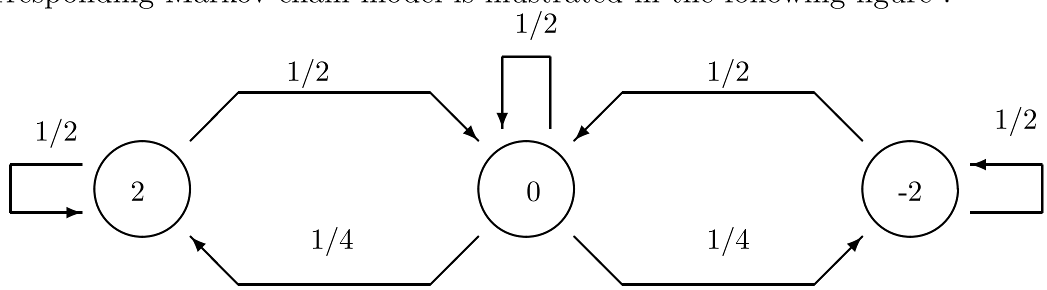
A plot of the power spectral density $\Phi_B(f)$ is given in the following figure :



(c) The transition matrix is :

I_{n-1}	I_n	B_n	I_{n+1}	B_{n+1}
-1	-1	-2	-1	-2
-1	-1	-2	1	0
-1	1	0	-1	0
-1	1	0	1	2
1	-1	0	-1	-2
1	-1	0	1	0
1	1	2	-1	0
1	1	2	1	2

The corresponding Markov chain model is illustrated in the following figure :



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Problem 4.22 :

(a) $I_n = a_n - a_{n-2}$, with the sequence $\{a_n\}$ being uncorrelated random variables (i.e $E(a_{n+m}a_n) = \delta(m)$). Hence :

$$\begin{aligned} \phi_{ii}(m) &= E[I_{n+m}I_n] = E[(a_{n+m} - a_{n+m-2})(a_n - a_{n-2})] \\ &= 2\delta(m) - \delta(m-2) - \delta(m+2) \\ &= \left\{ \begin{array}{cc} 2, & \text{m} = 0 \\ -1, & \text{m} = \pm 2 \\ 0, & \text{o.w.} \end{array} \right\} \end{aligned}$$

(b) $\Phi_{uu}(f) = \frac{1}{T}|G(f)|^2 \Phi_{ii}(f)$ where :

$$\begin{aligned} \Phi_{ii}(f) &= \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \exp(-j2\pi f m T) = 2 - \exp(j4\pi f T) - \exp(-j4\pi f T) \\ &= 2[1 - \cos 4\pi f T] = 4 \sin^2 2\pi f T \end{aligned}$$

and

$$|G(f)|^2 = (AT)^2 \left(\frac{\sin \pi f T}{\pi f T} \right)^2$$

Therefore :

$$\Phi_{uu}(f) = 4A^2T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 \sin{}^2 2\pi fT$$

(c) If $\{a_n\}$ takes the values (0,1) with equal probability then $E(a_n) = 1/2$ and $E(a_{n+m}a_n) = \left\{ \begin{array}{ll} 1/4, & \text{m} \neq 0 \\ 1/2, & \text{m} = 0 \end{array} \right\} = [1+\delta(m)]/4$. Then :

$$\begin{array}{rcl} \phi_{ii}(m) & = & E[I_{n+m}I_n] = 2\phi_{aa}(0) - \phi_{aa}(2) - \phi_{aa}(-2) \\ & = & \frac{1}{4}[2\delta(m) - \delta(m-2) - \delta(m+2)] \end{array}$$

and

$$\begin{array}{l} \Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m) \exp(-j2\pi fmT) = \sin{}^2 2\pi fT \\ \Phi_{uu}(f) = A^2T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 \sin{}^2 2\pi fT \end{array}$$

Thus, we obtain the same result as in (b) , but the magnitude of the various quantities is reduced by a factor of 4 .

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