Problem 5.1 :

(a) Taking the inverse Fourier transform of $H(f)$, we obtain :

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \text{sgn}(t) - \text{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right) \end{aligned}$$

where $sgn(x)$ is the signum signal (1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$) and $\Pi(x)$ is a rectangular pulse of unit height and width, centered at $x = 0$.

(b) The signal waveform, to which $h(t)$ is matched, is :

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 5.2 :

(a) The impulse response of the matched filter is :

$$h(t) = s(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

(b) The output of the matched filter at $t = T$ is :

$$\begin{aligned} g(T) &= h(t) \star s(t)|_{t=T} = \int_0^T h(T-\tau)s(\tau)d\tau \\ &= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau \\ &\overset{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2\cos^2(2\pi f_c v)dv \\ &= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\ &= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right] \end{aligned}$$

(c) The output of the correlator at $t = T$ is :

$$\begin{aligned} q(T) &= \int_0^T s^2(\tau)d\tau \\ &= \frac{A^2}{T^2}\int_0^T \tau^2 \cos^2(2\pi f_c \tau)d\tau \end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vice versa.

Problem 5.4 :

(a) The correlation type demodulator employes a filter :

$$f(t) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{T}} & 0 \leq \mathrm{t} \leq \mathrm{T} \\ 0 & \mathrm{o.w} \end{array} \right\}$$

as given in Example 5-1-1. Hence, the sampled outputs of the crosscorrelators are :

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance :

$$\sigma_n^2 \frac{N_0}{2}$$

The probability density function for the sampled output is :

$$\begin{aligned} p(r|s_0) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{r^2}{N_0}} \\ p(r|s_1) &= \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r - A\sqrt{T})^2}{N_0}} \end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\mathrm{PM}(\mathbf{r}, \mathbf{s}_0) = p(r|s_0) > p(r|s_1) = \mathrm{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as:

$$\frac{\text{PM}(\mathbf{r},\mathbf{s}_0)}{\text{PM}(\mathbf{r},\mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2-r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently :

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.

(b) The average probability of error is:

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} p(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} p(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\text{SNR}}\right]
\end{aligned}$$

where

$$\text{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

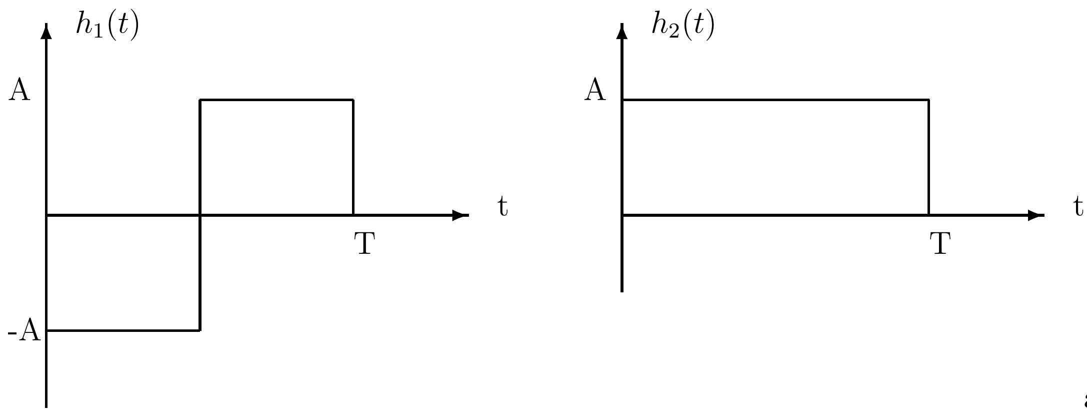
Problem 5.8 :

(a) Since the given waveforms are the equivalent lowpass signals :

$$\mathcal{E}_1 = \frac{1}{2}\int_0^T |s_1(t)|^2\, dt = \frac{1}{2}A^2 \int_0^T dt = A^2T/2$$
$$\mathcal{E}_2 = \frac{1}{2}\int_0^T |s_2(t)|^2\, dt = \frac{1}{2}A^2 \int_0^T dt = A^2T/2$$

Hence $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}$. Also :$\rho_{12} = \frac{1}{2\mathcal{E}} \int_0^T s_1(t) s_2^*(t) dt = 0$.

(b) Each matched filter has an equivalent lowpass impulse response : $h_i(t) = s_i(T-t)$. The following figure shows $h_i(t)$:



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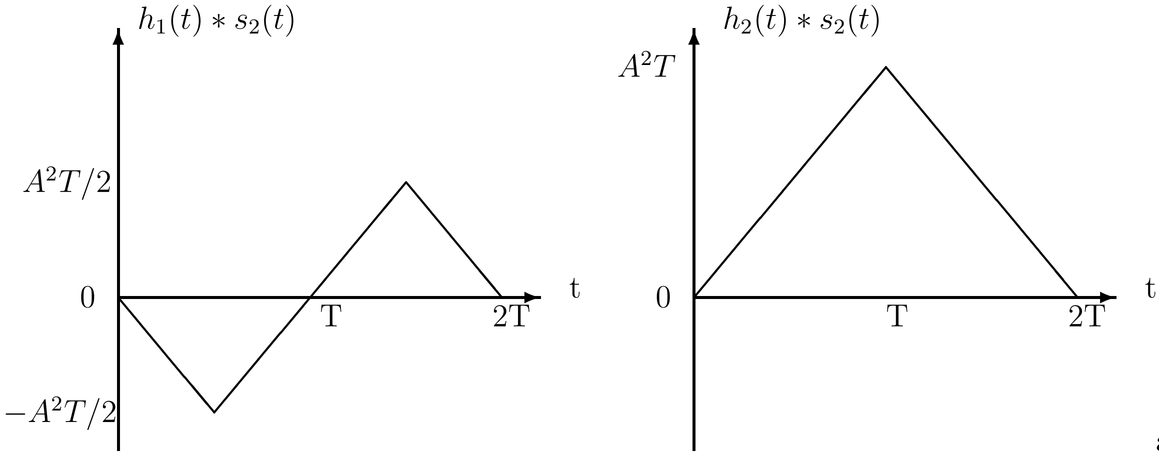
(c)



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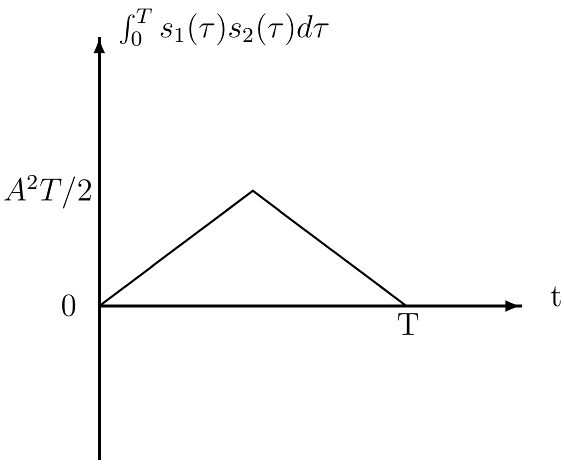
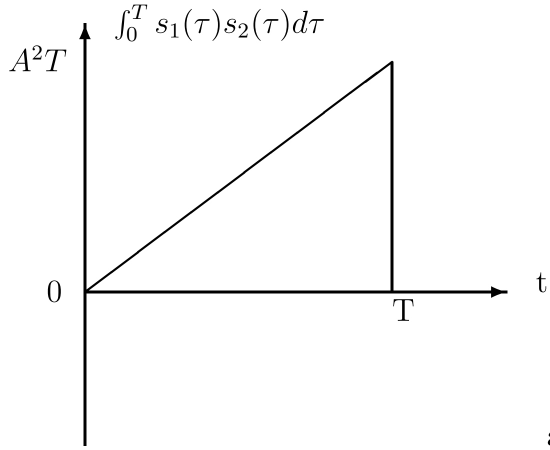
(d)





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(e) The outputs of the matched filters are different from the outputs of the correlators. The two sets of outputs agree at the sampling time $t = T$.

(f) Since the signals are orthogonal ($\rho_{12} = 0$) the error probability for AWGN is $P_2 = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)$, where $\mathcal{E} = A^2T/2$.

Problem 5.10 :

(a) $U = Re\left[\int_0^T r(t)s^*(t)dt\right]$, where $r(t) = \left\{\begin{array}{c} s(t)+z(t) \\ -s(t)+z(t) \\ z(t) \end{array}\right\}$ depending on which signal was sent. If we assume that $s(t)$ was sent :

$$U = Re\left[\int_0^T s(t)s^*(t)dt\right] + Re\left[\int_0^T z(t)s^*(t)dt\right] = 2E + N$$

where $E = \frac{1}{2}\int_0^T s(t)s^*(t)dt$, and $N = Re\left[\int_0^T z(t)s^*(t)dt\right]$ is a Gaussian random variable with zero mean and variance $2EN_0$ (as we have seen in Problem 5.7). Hence, given that $s(t)$ was sent, the probability of error is :

$$P_{e1} = P(2E + N < A) = P\left(N < -(2E - A)\right) = Q\left(\frac{2E - A}{\sqrt{2N_0 E}}\right)$$

When $-s(t)$ is transmitted : $U = -2E + N$, and the corresponding conditional error probability is :

$$P_{e2} = P(-2E + N > -A) = P\left(N > (2E - A)\right) = Q\left(\frac{2E - A}{\sqrt{2N_0 E}}\right)$$

and finally, when 0 is transmitted : $U = N$, and the corresponding error probability is :

$$P_{e3} = P(N > A \text{ or } N < -A) = 2P(N > A) = 2Q\left(\frac{A}{\sqrt{2N_0 E}}\right)$$

(b)

$$P_e = \frac{1}{3}\left(P_{e1} + P_{e2} + P_{e3}\right) = \frac{2}{3}\left[Q\left(\frac{2E - A}{\sqrt{2N_0 E}}\right) + Q\left(\frac{A}{\sqrt{2N_0 E}}\right)\right]$$

(c) In order to minimize P_e :

$$\frac{dP_e}{dA} = 0 \Rightarrow A = E$$

where we differentiate $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt$ with respect to x, using the Leibnitz rule : $\frac{d}{dx}\left(\int_{f(x)}^\infty g(a)da\right) = -\frac{df}{dx} g(f(x))$. Using this threshold :

$$P_e = \frac{4}{3} Q\left(\frac{E}{\sqrt{2N_0 E}}\right) = \frac{4}{3} Q\left(\sqrt{\frac{E}{2N_0}}\right)$$