Problem 5.12 :

The correlation of the two signals in binary FSK is:

$$\rho = \frac{\sin(2\pi\Delta f T)}{2\pi\Delta f T}$$

To find the minimum value of the correlation, we set the derivative of ρ with respect to Δf equal to zero. Thus:

$$\frac{\vartheta\rho}{\vartheta\Delta f} = 0 = \frac{\cos(2\pi\Delta f T)2\pi T}{2\pi\Delta f T} - \frac{\sin(2\pi\Delta f T)}{(2\pi\Delta f T)^2}2\pi T$$

and therefore :

$$2\pi\Delta f T = \tan(2\pi\Delta f T)$$

Solving numerically (or graphically) the equation $x = \tan(x)$, we obtain $x = 4.4934$. Thus,

$$2\pi\Delta f T = 4.4934 \Longrightarrow \Delta f = \frac{0.7151}{T}$$

and the value of ρ is -0.2172.
We know that the probability of error can be expressed in terms of the distance d_{12} between the signal points, as :

$$P_e = Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right]$$

where the distance between the two signal points is :

$$d_{12}^2 = 2\mathcal{E}_b(1-\rho)$$

and therefore :

$$P_e = Q\left[\sqrt{\frac{2\mathcal{E}_b(1-\rho)}{2N_0}}\right] = Q\left[\sqrt{\frac{1.2172\mathcal{E}_b}{N_0}}\right]$$

Problem 5.13 :

(a) It is straightforward to see that :

$$\begin{array}{l} \text{Set I : Four} - \text{level PAM} \\ \text{Set II : Orthogonal} \\ \text{Set III : Biorthogonal} \end{array}$$

(b) The transmitted waveforms in the first set have energy : $\frac{1}{2}A^2$ or $\frac{1}{2}9A^2$. Hence for the first set the average energy is :

$$\mathcal{E}_1 = \frac{1}{4}\left(2\frac{1}{2}A^2 + 2\frac{1}{2}9A^2\right) = 2.5A^2$$

All the waveforms in the second and third sets have the same energy : $\frac{1}{2}A^2$.Hence :

$$\mathcal{E}_2 = \mathcal{E}_3 = A^2/2$$

(c) The average probability of a symbol error for M-PAM is (5-2-45) :

$$P_{4,PAM} = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}}\right) = \frac{3}{2}Q\left(\sqrt{\frac{A^2}{N_0}}\right)$$

(d) For coherent detection, a union bound can be given by (5-2-25) :

$$P_{4,orth} < (M-1)\, Q\left(\sqrt{\mathcal{E}_s/N_0}\right) = 3Q\left(\sqrt{\frac{A^2}{2N_0}}\right)$$

while for non-coherent detection :

$$P_{4,orth,nc} \leq (M-1)\, P_{2,nc} = 3\frac{1}{2}e^{-\mathcal{E}_s/2N_0} = \frac{3}{2}e^{-A^2/4N_0}$$

(e) It is not possible to use non-coherent detection for a biorthogonal signal set : e.g. without phase knowledge, we cannot distinguish between the signals $u_1(t)$ and $u_3(t)$ (or $u_2(t)/u_4(t)$).

(f) The bit rate to bandwidth ratio for M-PAM is given by (5-2-85) :

$$\left(\frac{R}{W}\right)_1 = 2\log_2 M = 2\log_2 4 = 4$$

For orthogonal signals we can use the expression given by (5-2-86) or notice that we use a symbol interval 4 times larger than the one used in set I, resulting in a bit rate 4 times smaller :

$$\left(\frac{R}{W}\right)_2 = \frac{2\log_2 M}{M} = 1$$

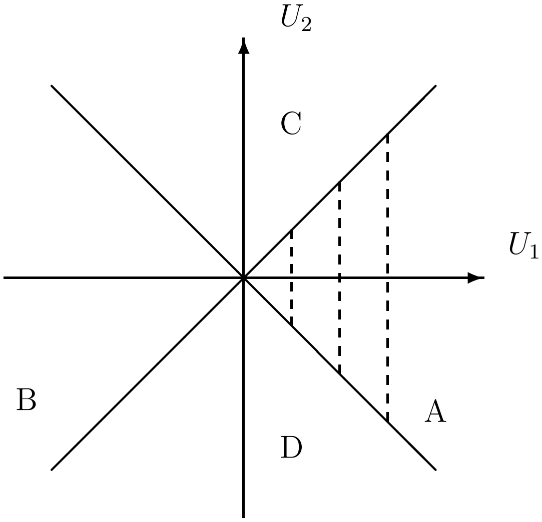
Finally, the biorthogonal set has double the bandwidth efficiency of the orthogonal set :

$$\left(\frac{R}{W}\right)_3 = 2$$

Hence, set I is the most bandwidth efficient (at the expense of larger average power), but set III will also be satisfactory.

Problem 5.14 :

The following graph shows the decision regions for the four signals :

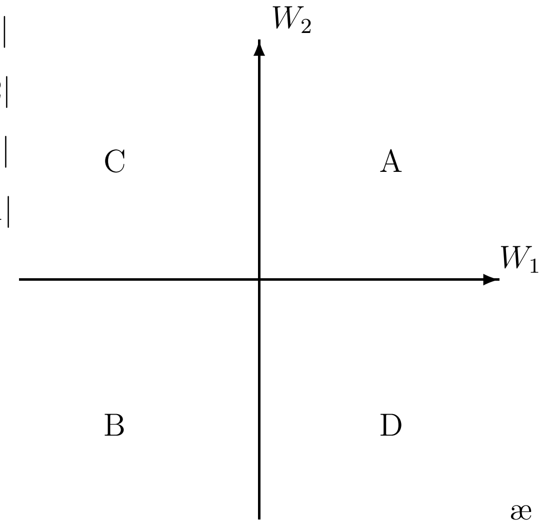


$A = U1 > +|U2|$

$B = U1 < -|U2|$

$C = U2 > +|U1|$

$D = U2 < -|U1|$



As we see, using the transformation $W_1 = U_1 + U_2,\ W_2 = U_1 - U_2$ alters the decision regions to : $(W_1 > 0, W_2 > 0 \rightarrow s_1(t); W_1 > 0, W_2 < 0 \rightarrow s_2(t);\ \text{etc.})$. Assuming that $s_1(t)$ was transmitted, the outputs of the matched filters will be :

$$\begin{array}{l} U_1 = 2\mathcal{E} + N_{1r} \\ U_2 = N_{2r} \end{array}$$

where N_{1r}, N_{2r} are uncorrelated (Prob. 5.7) Gaussian-distributed terms with zero mean and variance $2\mathcal{E}N_0$. Then :

$$\begin{array}{l} W_1 = 2\mathcal{E} + (N_{1r} + N_{2r}) \\ W_2 = 2\mathcal{E} + (N_{1r} - N_{2r}) \end{array}$$

will be Gaussian distributed with means : $E[W_1] = E[W_2] = 2\mathcal{E}$, and variances : $E[W_1^2] = E[W_2^2] = 4\mathcal{E}N_0$. Since U_1, U_2 are independent, it is straightforward to prove that W_1, W_2 are independent, too. Hence, the probability that a correct decision is made, assuming that $s_1(t)$ was transmitted is :

$$\begin{array}{rcl} P_{c|s1} & = & P[W_1 > 0]\,P[W_2 > 0] = (P[W_1 > 0])^2 \\ & = & (1 - P[W_1 < 0])^2 = \left(1 - Q\left(\frac{2\mathcal{E}}{\sqrt{4\mathcal{E}N_0}}\right)\right)^2 \\ & = & \left(1 - Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)\right)^2 = \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2 \end{array}$$

where $\mathcal{E}_b = \mathcal{E}/2$ is the transmitted energy per bit. Then :

$$P_{e|s1} = 1 - P_{c|s1} = 1 - \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2 = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]$$

This is the exact symbol error probability for the 4-PSK signal, which is expected since the vector space representations of the 4-biorthogonal and 4-PSK signals are identical.

Problem 5.15 :

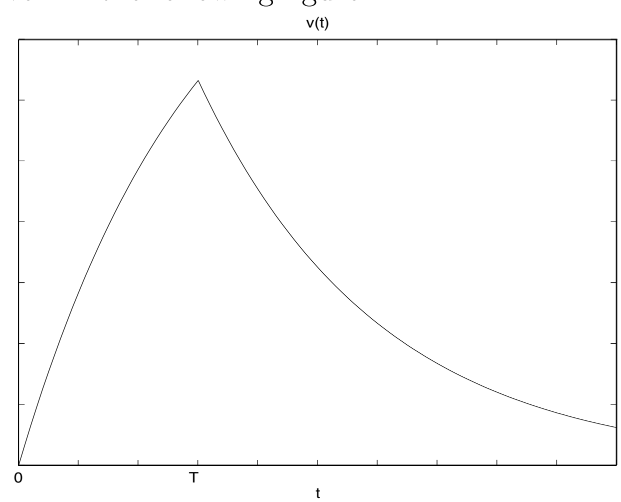
(a) The output of the matched filter can be expressed as :

$$y(t) = Re\left[v(t)e^{j2\pi f_c t}\right]$$

where $v(t)$ is the lowpass equivalent of the output :

$$v(t) = \int_0^t s_0(\tau)h(t-\tau)d\tau = \left\{ \begin{array}{ll} \int_0^t Ae^{-(t-\tau)/T}d\tau = AT\left(1-e^{-t/T}\right), & 0 \leq t \leq T \\ \int_0^T Ae^{-(t-\tau)/T}d\tau = AT(e-1)e^{-t/T}, & T \leq t \end{array} \right\}$$

(b) A sketch of $v(t)$ is given in the following figure :



(c) $y(t) = v(t)\cos 2\pi f_c t$, where $f_c >> 1/T$. Hence the maximum value of y corresponds to the maximum value of v, or $y_{\max} = y(T) = v_{\max} = v(T) = AT(1-e^{-1})$.

(d) Working with lowpass equivalent signals, the noise term at the sampling instant will be :

$$v_N(T) = \int_0^T z(\tau)h(T-\tau)d\tau$$

The mean is : $E\left[v_N(T)\right] = \int_0^T E\left[z(\tau)\right]h(T-\tau)d\tau = 0$, and the second moment :

$$\begin{array}{rcl} E\left[|v_N(T)|^2\right] & = & E\left[\int_0^T z(\tau)h(T-\tau)d\tau \int_0^T z^*(w)h(T-w)dw\right] \\ & = & 2N_0\int_0^T h^2(T-\tau)d\tau \\ & = & N_0 T\left(1-e^{-2}\right) \end{array}$$

The variance of the real-valued noise component can be obtained using the relationship $Re[N] = \frac{1}{2}(N + N^*)$ to obtain : $\sigma_{Nr}^2 = \frac{1}{2}E\left[|v_N(T)|^2\right] = \frac{1}{2}N_0T\left(1 - e^{-2}\right)$

(e) The SNR is defined as :

$$\gamma = \frac{|v_{\max}|^2}{E\left[|v_N(T)|^2\right]} = \frac{A^2T}{N_0}\frac{e-1}{e+1}$$

(the same result is obtained if we consider the real bandpass signal, when the energy term has the additional factor 1/2 compared to the lowpass energy term, and the noise term is $\sigma_{Nr}^2 = \frac{1}{2}E\left[|v_N(T)|^2\right]$)

(f) If we have a filter matched to $s_0(t)$, then the output of the noise-free matched filter will be :

$$v_{\max} = v(T) = \int_0^T s_o^2(t) = A^2T$$

and the noise term will have second moment :

$$\begin{aligned} E\left[|v_N(T)|^2\right] &= E\left[\int_0^T z(\tau)s_0(T-\tau)d\tau \int_0^T z^*(w)s_0(T-w)dw\right] \\ &= 2N_0\int_0^T s_0^2(T-\tau)d\tau \\ &= 2N_0A^2T \end{aligned}$$

giving an SNR of :

$$\gamma = \frac{|v_{\max}|^2}{E\left[|v_N(T)|^2\right]} = \frac{A^2T}{2N_0}$$

Compared with the result we obtained in (e), using a sub-optimum filter, the loss in SNR is equal to : $\left(\frac{e-1}{e+1}\right)\left(\frac{1}{2}\right)^{-1} = 0.925$ or approximately 0.35 dB

Problem 5.16 :

(a) Consider the QAM constellation of Fig. P5-16. Using the Pythagorean theorem we can find the radius of the inner circle as:

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain:

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2}A$$

(b) If we denote by r the radius of the circle, then using the cosine theorem we obtain:

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

(c) The average transmitted power of the PSK constellation is:

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation:

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is:

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

Problem 5.18 :

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 5.26 :

(a) The number of bits per symbol is

$$k = \frac{4800}{R} = \frac{4800}{2400} = 2$$

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with $M = 2^k$, is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3k\mathcal{E}_b}{(M-1)N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ and $k = 2$ we obtain

$$Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 5 \times 10^{-6} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 9.7682$$

(b) If the bit rate of transmission is 9600 bps, then

$$k = \frac{9600}{2400} = 4$$

In this case a 16-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{4}\right) Q\left[\sqrt{\frac{3 \times 4 \times \mathcal{E}_b}{15 \times N_0}}\right]\right)^2$$

Thus,

$$Q\left[\sqrt{\frac{3 \times \mathcal{E}_b}{15 \times N_0}}\right] = \frac{1}{3} \times 10^{-5} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 25.3688$$

(c) If the bit rate of transmission is 19200 bps, then

$$k = \frac{19200}{2400} = 8$$

In this case a 256-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{16}\right) Q\left[\sqrt{\frac{3 \times 8 \times \mathcal{E}_b}{255 \times N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ we obtain

$$\frac{\mathcal{E}_b}{N_0} = 659.8922$$

(d) The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

k	2	4	8
SNR (db)	9.89	14.04	28.19

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.

Problem 5.28 :

For 4-phase PSK ($M = 4$) we have the following realtionship between the symbol rate $1/T$, the required bandwith W and the bit rate $R = k \cdot 1/T = \frac{log_2 M}{T}$ (see 5-2-84):

$$W = \frac{R}{log_2 M} \rightarrow R = W log_2 M = 2W = 200 \text{ kbits/sec}$$

For binary FSK ($M = 2$) the required frequency separation is $1/2T$ (assuming coherent receiver) and (see 5-2-86):

$$W = \frac{M}{log_2 M} R \rightarrow R = \frac{2W log_2 M}{M} = W = 100 \text{ kbits/sec}$$

Finally, for 4-frequency non-coherent FSK, the required frequency separation is $1/T$, so the symbol rate is half that of binary coherent FSK, but since we have two bits/symbol, the bit ate is tha same as in binary FSK :

$$R = W = 100 \text{ kbits/sec}$$