EE 163
Communication Theory I

Winter 2003

http://ee163.caltech.edu

HW 6 Solutions

1. Consider the following detection problem:

$$r(t) = s_m(t) + n(t) \qquad 0 \le t \le T$$

for $m = 1, 2, \ldots, 8$, where

$$s_m(t) = \sqrt{2E/T}\sin\left(t + m\frac{\pi}{4}\right)$$

with E being the symbol energy.

This signal set is known as 8-PSK. Assume the 8 signals are equally likely, and $n(t)$ is AWGN with PSD $N_0/2$ watts/Hz.

(a) Choose a suitable orthonormal set for the signal space. Plot the signal constellation. What is the dimensionality of this space?

(b) Compute the distances (as a function of E) between an arbitrary signal point in the constellation, and the seven other points.

(c) Indicate the optimum decision regions on your signal constellation plot.

(d) Use the results from parts (b) and (c) to compute a union bound as an upper bound on the probability of symbol error.

(a) Dimensionality of this space is 2:

$$f_1(t) = \sqrt{\frac{2}{T}}\sin(t + \pi/4) \qquad f_2(t) = \sqrt{\frac{2}{T}}\cos(t + \pi/4) \qquad 0 \le t \le T$$

The constellation consists of 8 points around a circle of radius $\sqrt{E}$:

$s_1(t) = (\sqrt{E}, 0)$ (along the positive f_1 axis) $\qquad s_2(t) = (\sqrt{E/2}, \sqrt{E/2})$

$s_3(t) = (0, \sqrt{E})$ (along the positive f_2 axis) $\qquad s_4(t) = (-\sqrt{E/2}, \sqrt{E/2})$

$s_5(t) = (-\sqrt{E}, 0)$ (along the negative f_1 axis) $\qquad s_6(t) = (-\sqrt{E/2}, -\sqrt{E/2})$

$s_7(t) = (0, -\sqrt{E})$ (along the negative f_2 axis) $\qquad s_8(t) = (\sqrt{E/2}, -\sqrt{E/2})$

(b)

$d_{12}^2 = E/2 + (\sqrt{E} - \sqrt{E/2})^2$ or $d_{12} = d_{18} = [2E(1 - 1/\sqrt{2})]^{1/2}$

$d_{13} = d_{17} = \sqrt{2E}$

$d_{15} = 2\sqrt{E}$

$d_{14} = d_{16} = [2E(1 + 1/\sqrt{2})]^{1/2}$

(c) The decision regions are 45° pie-shaped regions, bordered on either side by radial lines from the constellation origin that bisect the arc between adjacent signal pairs.

(d)

$$P(\text{error}) \le 2Q\left[\sqrt{\frac{E}{N_0}\left(1 - \frac{1}{\sqrt{2}}\right)}\right] + 2Q\left[\sqrt{\frac{E}{N_0}}\right] + 2Q\left[\sqrt{\frac{E}{N_0}\left(1 + \frac{1}{\sqrt{2}}\right)}\right] + Q\left[\sqrt{\frac{2E}{N_0}}\right]$$

Problem 5.38 :

(a) The optimal ML detector (see 5-1-41) selects the sequence $\underline{C}_i$ that minimizes the quantity:

$$D(\mathbf{r}, \underline{C}_i) = \sum_{k=1}^{n} (r_k - \sqrt{\mathcal{E}_b} C_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$D(\mathbf{r}, \underline{C}_1) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2$$

and

$$D(\mathbf{r}, \underline{C}_2) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first term of the right side is common for the two equations, we conclude that the optimal ML detector can base its decisions only on the last $n - w$ received elements of $\mathbf{r}$. That is

$$\sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2 \ \underset{\underline{C}_1}{\overset{\underline{C}_2}{\gtrless}} \ 0$$

or equivalently

$$\sum_{k=w+1}^{n} r_k \ \underset{\underline{C}_2}{\overset{\underline{C}_1}{\gtrless}} \ 0$$

(b) Since $r_k = \sqrt{\mathcal{E}_b} C_{ik} + n_k$, the probability of error $P(e|\underline{C}_1)$ is

$$\begin{aligned} P(e|\underline{C}_1) &= P\left(\sqrt{\mathcal{E}_b}(n-w) + \sum_{k=w+1}^{n} n_k < 0 \right) \\ &= P\left(\sum_{k=w+1}^{n} n_k < -(n-w)\sqrt{\mathcal{E}_b} \right) \end{aligned}$$

The random variable $u = \sum_{k=w+1}^{n} n_k$ is zero-mean Gaussian with variance $\sigma_u^2 = (n-w)\sigma^2$. Hence

$$P(e|\underline{C}_1) = \frac{1}{\sqrt{2\pi(n-w)\sigma^2}} \int_{-\infty}^{-\sqrt{\mathcal{E}_b}(n-w)} \exp(-\frac{x^2}{2\pi(n-w)\sigma^2}) dx = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}} \right]$$

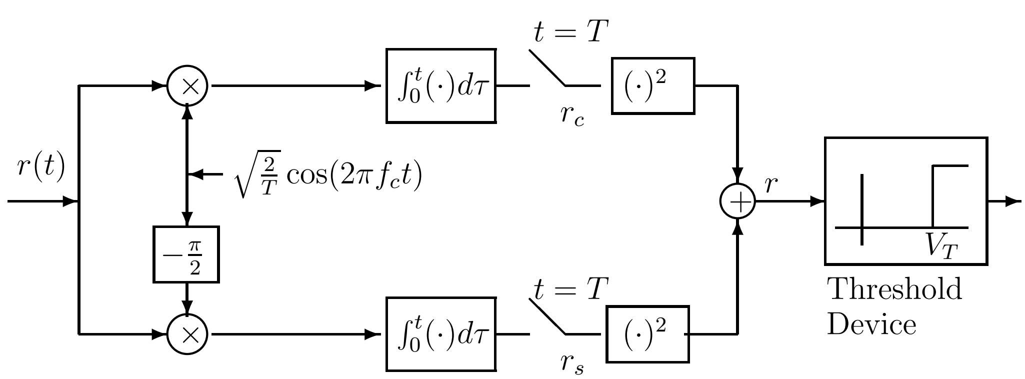
Similarly we find that $P(e|\underline{C}_2) = P(e|\underline{C}_1)$ and since the two sequences are equiprobable

$$P(e) = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

(c) The probability of error $P(e)$ is minimized when $\frac{\mathcal{E}_b(n-w)}{\sigma^2}$ is maximized, that is for $w = 0$. This implies that $\underline{C}_1 = -\underline{C}_2$ and thus the distance between the two sequences is the maximum possible.

Problem 5.42 :

(a) The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.



(b) If $s_0(t)$ is sent, then the received signal is $r(t) = n(t)$ and therefore the sampled outputs r_c, r_s are zero-mean independent Gaussian random variables with variance $\frac{N_0}{2}$. Hence, the random variable $r = \sqrt{r_c^2 + r_s^2}$ is Rayleigh distributed and the PDF is given by :

$$p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0} e^{-\frac{r^2}{N_0}}$$

If $s_1(t)$ is transmitted, then the received signal is :

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}} \cos(2\pi f_c t + \phi) + n(t)$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}} \cos(2\pi f_c t)$ and sampling the output at $t = T$, results in

$$\begin{aligned}
r_c &= \int_0^T r(t)\sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\
&= \int_0^T \frac{2\sqrt{\mathcal{E}_b}}{T_b} \cos(2\pi f_c t + \phi)\cos(2\pi f_c t) dt + \int_0^T n(t)\sqrt{\frac{2}{T}} \cos(2\pi f_c t) dt \\
&= \frac{2\sqrt{\mathcal{E}_b}}{T_b} \int_0^T \frac{1}{2} \left(\cos(2\pi 2 f_c t + \phi) + \cos(\phi)\right) dt + n_c \\
&= \sqrt{\mathcal{E}_b} \cos(\phi) + n_c
\end{aligned}$$

where n_c is zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Similarly, for the quadrature component we have :

$$r_s = \sqrt{\mathcal{E}_b} \sin(\phi) + n_s$$

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{\mathcal{E}_b + n_c^2 + n_s^2}$ follows the Rician distibution :

$$p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) = \frac{2r}{N_0} e^{-\frac{r^2+\mathcal{E}_b}{N_0}} I_0\left(\frac{2r\sqrt{\mathcal{E}_b}}{N_0}\right)$$

(c) For equiprobable signals the probability of error is given by:

$$P(\text{error}) = \frac{1}{2}\int_{-\infty}^{V_T} p(r|s_1(t))dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t))dr$$

Since $r > 0$ the expression for the probability of error takes the form

$$\begin{aligned}
P(\text{error}) &= \frac{1}{2}\int_0^{V_T} p(r|s_1(t))dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t))dr \\
&= \frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr + \frac{1}{2}\int_{V_T}^{\infty} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr
\end{aligned}$$

The optimum threshold level is the value of V_T that minimizes the probability of error. However, when $\frac{\mathcal{E}_b}{N_0} \gg 1$ the optimum value is close to: $\frac{\sqrt{\mathcal{E}_b}}{2}$ and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of $I_0(x)$ we will use the approximation :

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

which is valid for large x, that is for high SNR. In this case :

$$\frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2} e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) dr \approx \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{\mathcal{E}_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore :

$$\begin{aligned} \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr &\approx \frac{1}{2}\int_{-\infty}^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr \\ &= \frac{1}{2}Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] \end{aligned}$$

Finally :

$$\begin{aligned} P(\text{error}) &= \frac{1}{2}Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}\int_{\frac{\sqrt{\mathcal{E}_b}}{2}}^{\infty} \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} dr \\ &\leq \frac{1}{2}Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}e^{-\frac{\mathcal{E}_b}{4N_0}} \end{aligned}$$