

Problem 4.1 :

(a)

$$\hat{x}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(a)}{t-a} da$$

Hence :

$$\begin{aligned} -\hat{x}(-t) &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(a)}{-t-a} da \\ &= -\frac{1}{\pi} \int_{\infty}^{-\infty} \frac{x(-b)}{-t+b} (-db) \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(b)}{-t+b} db \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(b)}{t-b} db = \hat{x}(t) \end{aligned}$$

where we have made the change of variables : $b = -a$ and used the relationship : $x(b) = x(-b)$.

(b) In exactly the same way as in part (a) we prove :

$$\hat{x}(t) = \hat{x}(-t)$$

(c) $x(t) = \cos \omega_0 t$, so its Fourier transform is : $X(f) = \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)]$, $f_0 = 2\pi\omega_0$. Exploiting the phase-shifting property (4-1-7) of the Hilbert transform :

$$\hat{X}(f) = \frac{1}{2} [-j\delta(f - f_0) + j\delta(f + f_0)] = \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)] = F^{-1} \{\sin 2\pi f_0 t\}$$

Hence, $\hat{x}(t) = \sin \omega_0 t$.

(d) In a similar way to part (c) :

$$\begin{aligned} x(t) = \sin \omega_0 t &\Rightarrow X(f) = \frac{1}{2j} [\delta(f - f_0) - \delta(f + f_0)] \Rightarrow \hat{X}(f) = \frac{1}{2} [-\delta(f - f_0) - \delta(f + f_0)] \\ &\Rightarrow \hat{X}(f) = -\frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] = -F^{-1} \{\cos 2\pi\omega_0 t\} \Rightarrow \hat{x}(t) = -\cos \omega_0 t \end{aligned}$$

(e) The positive frequency content of the new signal will be : $(-j)(-j)X(f) = -X(f)$, $f > 0$, while the negative frequency content will be : $j \cdot jX(f) = -X(f)$, $f < 0$. Hence, since $\hat{\hat{X}}(f) = -X(f)$, we have : $\hat{\hat{x}}(t) = -x(t)$.

(f) Since the magnitude response of the Hilbert transformer is characterized by : $|H(f)| = 1$, we have that : $|\hat{X}(f)| = |H(f)||X(f)| = |X(f)|$. Hence :

$$\int_{-\infty}^{\infty} |\hat{X}(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df$$

and using Parseval's relationship :

$$\int_{-\infty}^{\infty} \hat{x}^2(t) dt = \int_{-\infty}^{\infty} x^2(t) dt$$

(g) From parts (a) and (b) above, we note that if $x(t)$ is even, $\hat{x}(t)$ is odd and vice-versa. Therefore, $x(t)\hat{x}(t)$ is always odd and hence : $\int_{-\infty}^{\infty} x(t)\hat{x}(t) dt = 0$.

Problem 4.2 :

We have :

$$\hat{x}(t) = h(t) * x(t)$$

where $h(t) = \frac{1}{\pi t}$ and $H(f) = \begin{cases} -j, & f > 0 \\ j, & f < 0 \end{cases}$. Hence :

$$\Phi_{\hat{x}\hat{x}}(f) = \Phi_{xx}(f) |H(f)|^2 = \Phi_{xx}(f)$$

and its inverse Fourier transform :

$$\phi_{\hat{x}\hat{x}}(\tau) = \phi_{xx}(\tau)$$

Also :

$$\begin{aligned} \phi_{x\hat{x}}(\tau) &= E[x(t+\tau)\hat{x}(t)] \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{E[x(t+\tau)x(a)]}{t-a} da \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi_{xx}(t+\tau-a)}{t-a} da \\ &= -\frac{1}{\pi} \int_{\infty}^{-\infty} \frac{\phi_{xx}(b)}{b-\tau} db \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\phi_{xx}(b)}{\tau-b} db = -\phi_{xx}(\tau) \end{aligned}$$

Problem 4.3 :

(a)

$$\begin{aligned} E[z(t)z(t+\tau)] &= E[\{x(t+\tau) + jy(t+\tau)\} \{x(t) + jy(t)\}] \\ &= E[x(t)x(t+\tau)] - E[y(t)y(t+\tau)] + jE[x(t)y(t+\tau)] \\ &\quad + E[y(t)x(t+\tau)] \\ &= \phi_{xx}(\tau) - \phi_{yy}(\tau) + j[\phi_{yx}(\tau) + \phi_{xy}(\tau)] \end{aligned}$$

But $\phi_{xx}(\tau) = \phi_{yy}(\tau)$ and $\phi_{yx}(\tau) = -\phi_{xy}(\tau)$. Therefore :

$$E[z(t)z(t+\tau)] = 0$$

(b)

$$\begin{aligned} V &= \int_0^T z(t)dt \\ E(V^2) &= \int_0^T \int_0^T E[z(a)z(b)] da db = 0 \end{aligned}$$

from the result in (a) above. Also :

$$\begin{aligned} E(VV^*) &= \int_0^T \int_0^T E[z(a)z^*(b)] da db \\ &= \int_0^T \int_0^T 2N_0\delta(a-b) da db \\ &= \int_0^T 2N_0 da = 2N_0T \end{aligned}$$

Problem 4.9 :

The energy of the signal waveform $s'_m(t)$ is :

$$\begin{aligned}\mathcal{E}' &= \int_{-\infty}^{\infty} |s'_m(t)|^2 dt = \int_{-\infty}^{\infty} \left| s_m(t) - \frac{1}{M} \sum_{k=1}^M s_k(t) \right|^2 dt \\ &= \int_{-\infty}^{\infty} s_m^2(t) dt + \frac{1}{M^2} \sum_{k=1}^M \sum_{l=1}^M \int_{-\infty}^{\infty} s_k(t) s_l(t) dt \\ &\quad - \frac{1}{M} \sum_{k=1}^M \int_{-\infty}^{\infty} s_m(t) s_k(t) dt - \frac{1}{M} \sum_{l=1}^M \int_{-\infty}^{\infty} s_m(t) s_l(t) dt \\ &= \mathcal{E} + \frac{1}{M^2} \sum_{k=1}^M \sum_{l=1}^M \mathcal{E} \delta_{kl} - \frac{2}{M} \mathcal{E} \\ &= \mathcal{E} + \frac{1}{M} \mathcal{E} - \frac{2}{M} \mathcal{E} = \left(\frac{M-1}{M} \right) \mathcal{E}\end{aligned}$$

The correlation coefficient is given by :

$$\begin{aligned}\rho_{mn} &= \frac{1}{\mathcal{E}'} \int_{-\infty}^{\infty} s'_m(t) s'_n(t) dt = \frac{1}{\mathcal{E}'} \int_{-\infty}^{\infty} \left(s_m(t) - \frac{1}{M} \sum_{k=1}^M s_k(t) \right) \left(s_n(t) - \frac{1}{M} \sum_{l=1}^M s_l(t) \right) dt \\ &= \frac{1}{\mathcal{E}'} \left(\int_{-\infty}^{\infty} s_m(t) s_n(t) dt + \frac{1}{M^2} \sum_{k=1}^M \sum_{l=1}^M \int_{-\infty}^{\infty} s_k(t) s_l(t) dt \right) \\ &\quad - \frac{1}{\mathcal{E}'} \left(\frac{1}{M} \sum_{k=1}^M \int_{-\infty}^{\infty} s_n(t) s_k(t) dt + \frac{1}{M} \sum_{l=1}^M \int_{-\infty}^{\infty} s_m(t) s_l(t) dt \right) \\ &= \frac{\frac{1}{M^2} M \mathcal{E} - \frac{1}{M} \mathcal{E} - \frac{1}{M} \mathcal{E}}{\frac{M-1}{M} \mathcal{E}} = -\frac{1}{M-1}\end{aligned}$$

Problem 4.11 :

(a) As an orthonormal set of basis functions we consider the set

$$\begin{aligned} f_1(t) &= \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{o.w} \end{cases} & f_2(t) &= \begin{cases} 1 & 1 \leq t < 2 \\ 0 & \text{o.w} \end{cases} \\ f_3(t) &= \begin{cases} 1 & 2 \leq t < 3 \\ 0 & \text{o.w} \end{cases} & f_4(t) &= \begin{cases} 1 & 3 \leq t < 4 \\ 0 & \text{o.w} \end{cases} \end{aligned}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

(b) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= [2 \quad -1 \quad -1 \quad -1] \\ \mathbf{s}_2 &= [-2 \quad 1 \quad 1 \quad 0] \\ \mathbf{s}_3 &= [1 \quad -1 \quad 1 \quad -1] \\ \mathbf{s}_4 &= [1 \quad -2 \quad -2 \quad 2] \end{aligned}$$

(c) The distance between the first and the second vector is:

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left| \begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix} \right|^2} = \sqrt{25}$$

Similarly we find that :

$$\begin{aligned} d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left| \begin{bmatrix} 1 & 0 & -2 & 0 \end{bmatrix} \right|^2} = \sqrt{5} \\ d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix} \right|^2} = \sqrt{12} \\ d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left| \begin{bmatrix} -3 & 2 & 0 & 1 \end{bmatrix} \right|^2} = \sqrt{14} \\ d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} -3 & 3 & 3 & -2 \end{bmatrix} \right|^2} = \sqrt{31} \\ d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left| \begin{bmatrix} 0 & 1 & 3 & -3 \end{bmatrix} \right|^2} = \sqrt{19} \end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 4.13 :

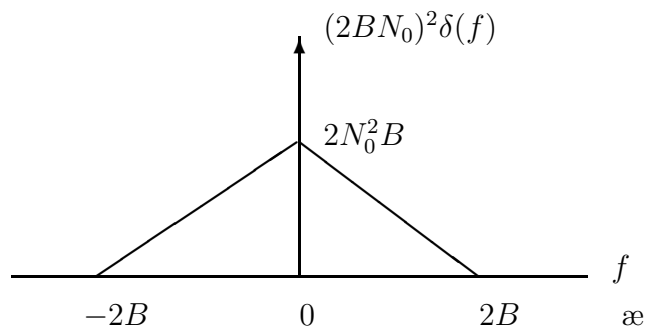
The power spectral density of $X(t)$ corresponds to : $\phi_{xx}(t) = 2BN_0 \frac{\sin 2\pi Bt}{2\pi Bt}$. From the result of Problem 2.14 :

$$\phi_{yy}(\tau) = \phi_{xx}^2(0) + 2\phi_{xx}^2(\tau) = (2BN_0)^2 + 8B^2 N_0^2 \left(\frac{\sin 2\pi Bt}{2\pi Bt} \right)^2$$

Also :

$$\Phi_{yy}(f) = \phi_{xx}^2(0)\delta(f) + 2\Phi_{xx}(f) * \Phi_{xx}(f)$$

The following figure shows the power spectral density of $Y(t)$:



Problem 4.17 :

The first basis function is :

$$g_4(t) = \frac{s_4(t)}{\sqrt{\mathcal{E}_4}} = \frac{s_4(t)}{\sqrt{3}} = \begin{cases} -1/\sqrt{3}, & 0 \leq t \leq 3 \\ 0, & \text{o.w.} \end{cases}$$

Then, for the second basis function :

$$c_{43} = \int_{-\infty}^{\infty} s_3(t)g_4(t)dt = -1/\sqrt{3} \Rightarrow g'_3(t) = s_3(t) - c_{43}g_4(t) = \begin{cases} 2/3, & 0 \leq t \leq 2 \\ -4/3, & 2 \leq t \leq 3 \\ 0, & \text{o.w} \end{cases}$$

Hence :

$$g_3(t) = \frac{g'_3(t)}{\sqrt{E_3}} = \begin{cases} 1/\sqrt{6}, & 0 \leq t \leq 2 \\ -2/\sqrt{6}, & 2 \leq t \leq 3 \\ 0, & \text{o.w} \end{cases}$$

where E_3 denotes the energy of $g'_3(t)$: $E_3 = \int_0^3 (g'_3(t))^2 dt = 8/3$.

For the third basis function :

$$c_{42} = \int_{-\infty}^{\infty} s_2(t)g_4(t)dt = 0 \quad \text{and} \quad c_{32} = \int_{-\infty}^{\infty} s_2(t)g_3(t)dt = 0$$

Hence :

$$g'_2(t) = s_2(t) - c_{42}g_4(t) - c_{32}g_3(t) = s_2(t)$$

and

$$g_2(t) = \frac{g'_2(t)}{\sqrt{\mathcal{E}_2}} = \begin{cases} 1/\sqrt{2}, & 0 \leq t \leq 1 \\ -1/\sqrt{2}, & 1 \leq t \leq 2 \\ 0, & \text{o.w} \end{cases}$$

where : $\mathcal{E}_2 = \int_0^2 (s_2(t))^2 dt = 2$.

Finally for the fourth basis function :

$$c_{41} = \int_{-\infty}^{\infty} s_1(t)g_4(t)dt = -2/\sqrt{3}, \quad c_{31} = \int_{-\infty}^{\infty} s_1(t)g_3(t)dt = 2/\sqrt{6}, \quad c_{21} = 0$$

Hence :

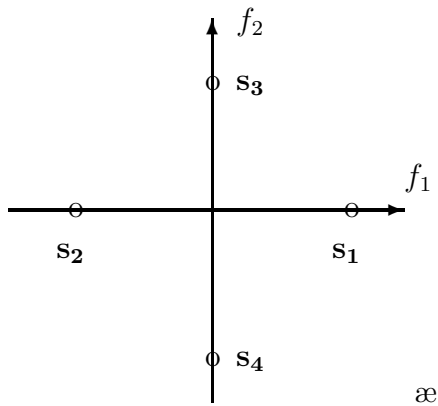
$$g'_1(t) = s_1(t) - c_{41}g_4(t) - c_{31}g_3(t) - c_{21}g_2(t) = 0 \Rightarrow g_1(t) = 0$$

The last result is expected, since the dimensionality of the vector space generated by these signals is 3. Based on the basis functions $(g_2(t), g_3(t), g_4(t))$ the basis representation of the signals is :

$$\begin{aligned}\mathbf{s}_4 &= (0, 0, \sqrt{3}) \Rightarrow \mathcal{E}_4 = 3 \\ \mathbf{s}_3 &= (0, \sqrt{8/3}, -1/\sqrt{3}) \Rightarrow \mathcal{E}_3 = 3 \\ \mathbf{s}_2 &= (\sqrt{2}, 0, 0) \Rightarrow \mathcal{E}_2 = 2 \\ \mathbf{s}_1 &= (2/\sqrt{6}, -2/\sqrt{3}, 0) \Rightarrow \mathcal{E}_1 = 2\end{aligned}$$

Problem 4.18 :

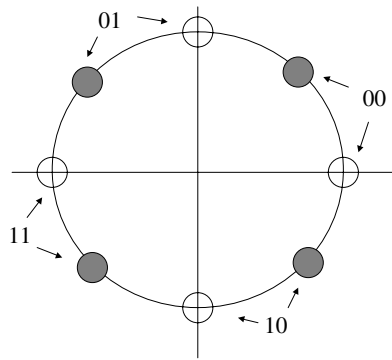
$$\begin{aligned} \mathbf{s}_1 &= (\sqrt{\mathcal{E}}, 0) \\ \mathbf{s}_2 &= (-\sqrt{\mathcal{E}}, 0) \\ \mathbf{s}_3 &= (0, \sqrt{\mathcal{E}}) \\ \mathbf{s}_4 &= (0, -\sqrt{\mathcal{E}}) \end{aligned}$$



As we see, this signal set is indeed equivalent to a 4-phase PSK signal.

Problem 4.19 :

(a)(b) The signal space diagram, together with the Gray encoding of each signal point is given in the following figure :



The signal points that may be transmitted at times $t = 2nT$ $n = 0, 1, \dots$ are given with blank circles, while the ones that may be transmitted at times $t = 2nT + 1$, $n = 0, 1, \dots$ are given with filled circles.