

Problem 5.12 :

The correlation of the two signals in binary FSK is:

$$\rho = \frac{\sin(2\pi\Delta fT)}{2\pi\Delta fT}$$

To find the minimum value of the correlation, we set the derivative of ρ with respect to Δf equal to zero. Thus:

$$\frac{\partial \rho}{\partial \Delta f} = 0 = \frac{\cos(2\pi\Delta fT)2\pi T}{2\pi\Delta fT} - \frac{\sin(2\pi\Delta fT)}{(2\pi\Delta fT)^2}2\pi T$$

and therefore :

$$2\pi\Delta fT = \tan(2\pi\Delta fT)$$

Solving numerically (or graphically) the equation $x = \tan(x)$, we obtain $x = 4.4934$. Thus,

$$2\pi\Delta fT = 4.4934 \implies \Delta f = \frac{0.7151}{T}$$

and the value of ρ is -0.2172 .

We know that the probability of error can be expressed in terms of the distance d_{12} between the signal points, as :

$$P_e = Q \left[\sqrt{\frac{d_{12}^2}{2N_0}} \right]$$

where the distance between the two signal points is :

$$d_{12}^2 = 2\mathcal{E}_b(1 - \rho)$$

and therefore :

$$P_e = Q \left[\sqrt{\frac{2\mathcal{E}_b(1 - \rho)}{2N_0}} \right] = Q \left[\sqrt{\frac{1.2172\mathcal{E}_b}{N_0}} \right]$$

Problem 5.13 :

(a) It is straightforward to see that :

- Set I : Four – level PAM
- Set II : Orthogonal
- Set III : Biorthogonal

(b) The transmitted waveforms in the first set have energy : $\frac{1}{2}A^2$ or $\frac{1}{2}9A^2$. Hence for the first set the average energy is :

$$\mathcal{E}_1 = \frac{1}{4} \left(2\frac{1}{2}A^2 + 2\frac{1}{2}9A^2 \right) = 2.5A^2$$

All the waveforms in the second and third sets have the same energy : $\frac{1}{2}A^2$.Hence :

$$\mathcal{E}_2 = \mathcal{E}_3 = A^2/2$$

(c) The average probability of a symbol error for M-PAM is (5-2-45) :

$$P_{4,PAM} = \frac{2(M-1)}{M} Q \left(\sqrt{\frac{6\mathcal{E}_{av}}{(M^2-1)N_0}} \right) = \frac{3}{2} Q \left(\sqrt{\frac{A^2}{N_0}} \right)$$

(d) For coherent detection, a union bound can be given by (5-2-25) :

$$P_{4,orth} < (M - 1) Q \left(\sqrt{\mathcal{E}_s/N_0} \right) = 3Q \left(\sqrt{\frac{A^2}{2N_0}} \right)$$

while for non-coherent detection :

$$P_{4,orth,nc} \leq (M - 1) P_{2,nc} = 3 \frac{1}{2} e^{-\mathcal{E}_s/2N_0} = \frac{3}{2} e^{-A^2/4N_0}$$

(e) It is not possible to use non-coherent detection for a biorthogonal signal set : e.g. without phase knowledge, we cannot distinguish between the signals $u_1(t)$ and $u_3(t)$ (or $u_2(t)/u_4(t)$).

(f) The bit rate to bandwidth ratio for M-PAM is given by (5-2-85) :

$$\left(\frac{R}{W} \right)_1 = 2 \log_2 M = 2 \log_2 4 = 4$$

For orthogonal signals we can use the expression given by (5-2-86) or notice that we use a symbol interval 4 times larger than the one used in set I, resulting in a bit rate 4 times smaller :

$$\left(\frac{R}{W} \right)_2 = \frac{2 \log_2 M}{M} = 1$$

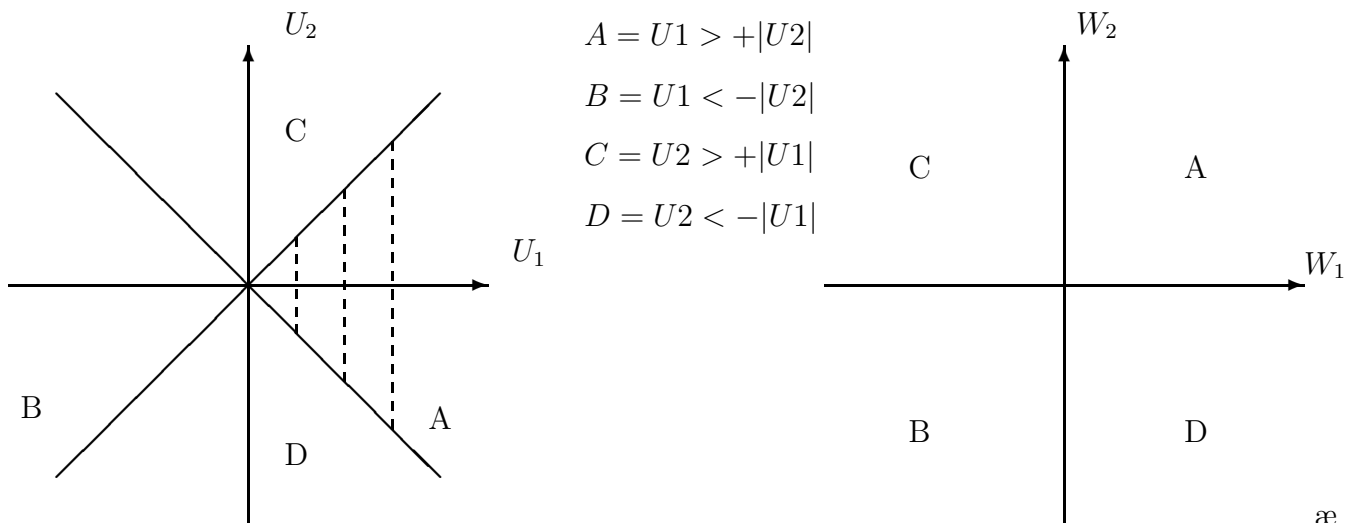
Finally, the biorthogonal set has double the bandwidth efficiency of the orthogonal set :

$$\left(\frac{R}{W} \right)_3 = 2$$

Hence, set I is the most bandwidth efficient (at the expense of larger average power), but set III will also be satisfactory.

Problem 5.14 :

The following graph shows the decision regions for the four signals :



As we see, using the transformation $W_1 = U_1 + U_2$, $W_2 = U_1 - U_2$ alters the decision regions to : ($W_1 > 0, W_2 > 0 \rightarrow s_1(t)$; $W_1 > 0, W_2 < 0 \rightarrow s_2(t)$; etc.). Assuming that $s_1(t)$ was transmitted, the outputs of the matched filters will be :

$$\begin{aligned} U_1 &= 2\mathcal{E} + N_{1r} \\ U_2 &= N_{2r} \end{aligned}$$

where N_{1r}, N_{2r} are uncorrelated (Prob. 5.7) Gaussian-distributed terms with zero mean and variance $2\mathcal{E}N_0$. Then :

$$\begin{aligned} W_1 &= 2\mathcal{E} + (N_{1r} + N_{2r}) \\ W_2 &= 2\mathcal{E} + (N_{1r} - N_{2r}) \end{aligned}$$

will be Gaussian distributed with means : $E[W_1] = E[W_2] = 2\mathcal{E}$, and variances : $E[W_1^2] = E[W_2^2] = 4\mathcal{E}N_0$. Since U_1, U_2 are independent, it is straightforward to prove that W_1, W_2 are independent, too. Hence, the probability that a correct decision is made, assuming that $s_1(t)$ was transmitted is :

$$\begin{aligned} P_{c|s_1} &= P[W_1 > 0] P[W_2 > 0] = (P[W_1 > 0])^2 \\ &= (1 - P[W_1 < 0])^2 = \left(1 - Q\left(\frac{2\mathcal{E}}{\sqrt{4\mathcal{E}N_0}}\right)\right)^2 \\ &= \left(1 - Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right)\right)^2 = \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2 \end{aligned}$$

where $\mathcal{E}_b = \mathcal{E}/2$ is the transmitted energy per bit. Then :

$$P_{e|s_1} = 1 - P_{c|s_1} = 1 - \left(1 - Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right)^2 = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) \left[1 - \frac{1}{2}Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)\right]$$

This is the exact symbol error probability for the 4-PSK signal, which is expected since the vector space representations of the 4-biorthogonal and 4-PSK signals are identical.

Problem 5.15 :

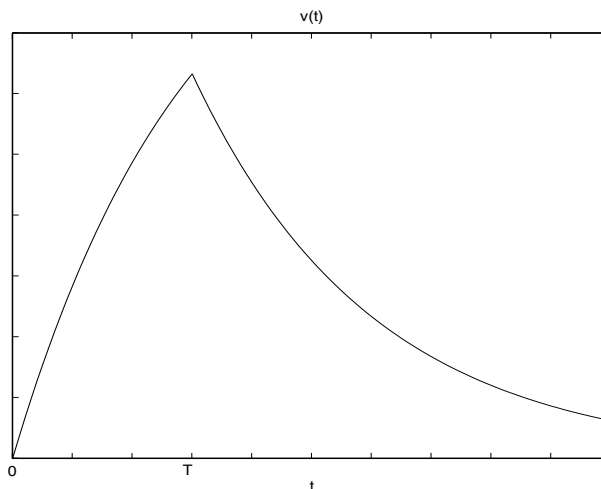
(a) The output of the matched filter can be expressed as :

$$y(t) = \text{Re} \left[v(t) e^{j2\pi f_c t} \right]$$

where $v(t)$ is the lowpass equivalent of the output :

$$v(t) = \int_0^t s_0(\tau) h(t - \tau) d\tau = \begin{cases} \int_0^t A e^{-(t-\tau)/T} d\tau = AT (1 - e^{-t/T}), & 0 \leq t \leq T \\ \int_0^T A e^{-(t-\tau)/T} d\tau = AT(e - 1)e^{-t/T}, & T \leq t \end{cases}$$

(b) A sketch of $v(t)$ is given in the following figure :



(c) $y(t) = v(t) \cos 2\pi f_c t$, where $f_c \gg 1/T$. Hence the maximum value of y corresponds to the maximum value of v , or $y_{\max} = y(T) = v_{\max} = v(T) = AT(1 - e^{-1})$.

(d) Working with lowpass equivalent signals, the noise term at the sampling instant will be :

$$v_N(T) = \int_0^T z(\tau) h(T - \tau) d\tau$$

The mean is : $E [v_N(T)] = \int_0^T E [z(\tau)] h(T - \tau) d\tau = 0$, and the second moment :

$$\begin{aligned} E [|v_N(T)|^2] &= E \left[\int_0^T z(\tau) h(T - \tau) d\tau \int_0^T z^*(w) h(T - w) dw \right] \\ &= 2N_0 \int_0^T h^2(T - \tau) d\tau \\ &= N_0 T (1 - e^{-2}) \end{aligned}$$

The variance of the real-valued noise component can be obtained using the relationship $Re[N] = \frac{1}{2}(N + N^*)$ to obtain : $\sigma_{Nr}^2 = \frac{1}{2}E[|v_N(T)|^2] = \frac{1}{2}N_0T(1 - e^{-2})$

(e) The SNR is defined as :

$$\gamma = \frac{|v_{\max}|^2}{E[|v_N(T)|^2]} = \frac{A^2T e - 1}{N_0 e + 1}$$

(the same result is obtained if we consider the real bandpass signal, when the energy term has the additional factor 1/2 compared to the lowpass energy term, and the noise term is $\sigma_{Nr}^2 = \frac{1}{2}E[|v_N(T)|^2]$)

(f) If we have a filter matched to $s_0(t)$, then the output of the noise-free matched filter will be :

$$v_{\max} = v(T) = \int_0^T s_o^2(t) dt = A^2T$$

and the noise term will have second moment :

$$\begin{aligned} E[|v_N(T)|^2] &= E\left[\int_0^T z(\tau)s_0(T-\tau)d\tau \int_0^T z^*(w)s_0(T-w)dw\right] \\ &= 2N_0 \int_0^T s_0^2(T-\tau)d\tau \\ &= 2N_0A^2T \end{aligned}$$

giving an SNR of :

$$\gamma = \frac{|v_{\max}|^2}{E[|v_N(T)|^2]} = \frac{A^2T}{2N_0}$$

Compared with the result we obtained in (e), using a sub-optimum filter, the loss in SNR is equal to : $\left(\frac{e-1}{e+1}\right)\left(\frac{1}{2}\right)^{-1} = 0.925$ or approximately 0.35 dB