California Institute of Technology
Department of Electrical Engineering

EE 163A
Communication Theory I

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http://ee163.caltech.edu

Midterm Exam 2-3-2005
Due: Monday 2-7-2005 in class

This is an open book and notes, but closed friends, family, and neighbors exam. You are permitted to use online resources, as long as it does not involve requests for help of any type, through email, chat rooms, instant messaging, etc, from other actual human beings (except me, the instructor). You do not need to sit through the exam at once, and can take breaks, go out, eat, sleep, watch Saturday Night Live, …. as long as the total amount of time you spend on solving the problems is less than or around 6-7 hours.

All the work in this exam is mine alone. I have neither received assistance from another person or group, nor have I given assistance to another person.

Name: ________________________________ **Signature/Date:** ________________________________

1. (20) Consider the design of a matched filter $H(f)$ for a binary input signal which contains *intersymbol interference* (ISI) introduced by the communication channel. In particular, assume that the ISI extends only back to the previous symbol. Mathematically speaking, let the input signal $s(t)$ be of the form

$$s(t) = a_0 p(t) + a_{-1} p(t+T)$$

where a_0 and a_{-1} are independent ± 1 data bits, and T is the bit interval ($1/T$ is the data/bit rate). Due to ISI $p(t)$ extends over 2-bit intervals, i.e., from $t = 0$ to $t = 2T$.

(a) Under the hypothesis that $a_0 = 1$, write expressions for the signal-to-noise ratio (SNR) at the output of the matched filter $H(f)$ at time $t = 2T$ under the two hypotheses $a_{-1} = -1$ and $a_{-1} = +1$. Express your answers in terms of the two-sided noise spectral density $\Phi_{nn}(f)$ and the Fourier transform of the pulse shape, namely, $P(f)$.

(b) Write expressions for the matched filter transfer function $H(f)$ under each of the two hypotheses $a_{-1} = -1$ and $a_{-1} = +1$.

(c) Simplify your answers in parts (a) and (b) for the case of white noise, i.e., $\Phi_{nn}(f) = N_0/2$.

(d) For the white noise case of part (c), show an implementation of the matched filter in terms of the matched filter $H_0(f)$ for a single pulse $p(t)$, i.e., the filter that maximizes the SNR at time $t = T$ when the input is $p(t) + n(t)$. Here, instead of considering the two hypotheses $a_{-1} = -1$ and $a_{-1} = +1$, assume that the value of a_{-1} is "known" and thus can be treated as a "constant". As such you will have one configuration whose implementation depends on a_{-1}. Show the mathematical expression for this implementation and illustrate it with a block diagram.

(e) If now a continuous data stream

$$m(t) = \sum_{n=-\infty}^{\infty} a_k p(t - nT)$$

is transmitted with $p(t)$ having the property as above, i.e., it is nonzero only over $(0, 2T)$, draw the receiver implementation that minimizes the error probability under the hypothesis that the previous bit is "known" as assumed in part (d). There is no need to derive the mathematical expression for the decision threshold.

2. (20) A linear filter $H(f)$ is designed to be a matched filter for a rectangular time-limited pulse

$$s(t) = \begin{cases} \sqrt{P} & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

Now, a triangular signal pulse

$$s(t) = \begin{cases} 2At/T & 0 \le t \le T/2 \\ -(2A/T)(t-T) & T/2 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

is applied to the matched filter together with the usual additive white Gaussian noise $n(t)$.

(a) Compute the signal-to-noise ratio of the matched filter output at time $t = T$. Express your answer only in terms of P, N_0, and T, where N_0 is the single-sided power spectral density of $n(t)$. Assume that the peak, A, of the triangular pulse is chosen such that its average power equals that of the rectangular pulse.

(b) How does the SNR computed in part (a) compare with that which would be achieved if the *rectangular* pulse $s(t)$ were applied to the matched filter input together with $n(t)$? Is there a gain or loss in SNR performance when the triangular pulse is used and by how much?

Suppose now that the problem was reversed, i.e., the filter is designed to be matched to the triangular pulse.

(c) What is the SNR of the matched filter output at time $t = T$ when the sum of $n(t)$ and the rectnagular pulse is applied to its input?

(d) Analogous to part (b), how does the SNR computed in part (c) compare with that which would be achieved if the *triangular* pulse $s(t)$ were applied to the mathced filter input together with $n(t)$? Again express your answer only in terms of P, N_0, and T. How does the gain or loss in this part compare with that computed for the reverse problem in part (b)?

3. (20) Consider the mathematical model of a discrete communication channel having 3 possible input messages $\{a, b, c\}$ and 3 possible output symbols $\{1, 2, 3\}$. The channel model is completely described by the set of nine conditional probabilities

$$\begin{array}{lll} P(1|a) = 0.6 & P(2|a) = 0.3 & P(3|a) = 0.1 \\ P(1|b) = 0.1 & P(2|b) = 0.5 & P(3|b) = 0.4 \\ P(1|c) = 0.1 & P(2|c) = 0.1 & P(3|c) = 0.8 \end{array}$$

which specify the probability of receiving each output symbol conditioned on each input message. Assume that we know the set of three a priori probabilities with which the input messages are transmitted, namely,

$$P(a) = 0.3, \;\; P(b) = 0.5, \;\text{ and } P(c) = 0.2$$

(a) Given that the received (channel output) symbol is 1, what is the decision of the optimum (maximum a posteriori) receiver regarding which message is transmitted?

(b) Repeat part (a) when the received symbol is 2.

(c) Repeat part (a) when the received symbol is 3.

(d) Does the mathematical model assumed for the channel, together with the given a priori probabilities, lead to a decision rule in which each of the possible transmitted signals (a, b, and c) can at some time be selected? If your answer is "no", then suggest a set of a priori probabilities that would make the above answer be "yes".

4. (20) A binary communication system is designed to decide between two hypotheses H_0 and H_1 where H_0 corresponds to the transmission of $s_0(t) + n(t)$ and H_1 corresponds to the transmission of $s_1(t) + n(t)$. Assume that $n(t)$ is a bandlimited white Gaussian noise process with single-sided noise spectral density N_0 and single-sided bandwidth B. The decision between H_0 and H_1 is to be made based on N *independent* observations (samples) of the received signal $y(t) = s_i(t) + n(t)$ taken in the signaling interval of duration T. Assume that $s_0(t)$ and $s_1(t)$ are rectangular pulses of duration T and amplitude $\sqrt{P}$, and are assumed to be transmitted with probabilities $p(H_0)$ and $p(H_1)$, respectively. Also, let y_i, s_{0i}, s_{1i}, and n_i; $n = 1, 2, \ldots N$ respectively denote the samples of $y(t)$, $s_0(t)$, $s_1(t)$, and $n(t)$ at time $t = (i - 1/2)T_s$ where $1/T_s$ denotes the sampling rate.

(a) What is the minimum value of N such that the samples of the received signal $y(t)$ are independent? Express your answer in terms of B and T.

(b) At each sample time, y_i is compared to a threshold γ and the following *per sample* correct decision probabilities are defined:

$$\begin{aligned} p &= \Pr\{H_1^{'}|H_1\} = \Pr\{y_i > \gamma|H_1\} \\ &= \Pr\{H_0^{'}|H_0\} = \Pr\{y_i < \gamma|H_0\} \end{aligned}$$

where the prime denotes the fact that the decision is based on only that sample. The final decision on which hypothesis is true (i.e., was $s_0(t)$ or $s_1(t)$ sent) is made according to the following rule:

Decide H_1 if there are λ or more samples, y_i, that exceed the threshold γ.
Decide H_0 if there are fewer than λ samples, y_i, that exceed γ.

Write an expression for the two conditional error probabilities, $\Pr\{H_1|H_0\}$, and $\Pr\{H_0|H_1\}$.

(c) Write an expression for the average probability of error.

(d) Suppose now that rather than making tentative decisions ($H_0^{'}$ or $H_1^{'}$) at each sample time, the N independent [assume the minimum value of N found in part (a)] samples y_i; $i = 1, 2, \ldots, N$ are merely summed and a final decision is made as follows:

Decide H_1 if the sum of all N samples, y_i, exceeds the threshold Λ.
Decide H_0 if the sum of all N samples, y_i, does not exceed the threshold Λ.

Write an expression for the average probability of error in this case. Express your answer in terms of P, T, N_0, N, Λ, and the a priori probabilities $p(H_0)$ and $p(H_1)$.

(e) How should Λ be chosen so as to make the conditional probabilities $\Pr\{H_1|H_0\}$, and $\Pr\{H_0|H_1\}$ equal?

5. (20) Consider a binary communication system described by two hypotheses H_0 and H_1 with a priori probabilities $p(H_0) = p$ and $p(H_1) = 1 - p$. In many situations the *cost* of making an error of the first kind, i.e., deciding H_1 when H_0 is indeed true, is not equal to the cost of making an error of the second kind, i.e., deciding H_0 when H_1 is true. In such cases, minimization of the *average cost* is a more appropriate optimization criterion than minimization of the *average error probability.*

In the above binary situation there are four possible combinations of hypotheses and decisions. We assign a cost to each of these four combinations. In particular, let C_{ij} denote the cost associated with choosing hypothesis H_i when indeed H_j is true for $i, j = 0, 1$ (note that in many systems the costs C_{00} and C_{11} assigned to correct decisions are often set equal to zero. For the moment, we will not make that assumption).

(a) Letting P_{ij} denote the conditional probability of choosing H_i given that H_j is true, write an expression for the average cost, $\bar{C}$. Express your answer only in terms of p, P_{10}, P_{01}, and $\{C_{ij}; i, j = 0, 1\}$.

(b) Assuming that the channel is modeled as a continuous memoryless channel, it is desired to implement the decision rule using a threshold test based on a single sample of the channel output, r. That is

Choose H_0 if $r > \gamma$
Choose H_1 if $r \leq \gamma$

where γ is a threshold to be determined. Letting $p_i(r) = p(r|H_i)$ denote the conditional probability density function of r given the hypothesis $H_i, i = 0, 1$, derive a condition that the threshold γ must satisfy to minimize the average cost $\bar{C}$ found in part (a). Express your answer only in terms of p, $\{p_i(r); i = 0, 1\}$, and $\{C_{ij}; i, j = 0, 1\}$. This minimum average cost criterion (often referred to as minimum *risk*) is called the Bayes criterion and the corresponding detector (decoder) is called a *Bayes detector.*

(c) Express the decision rule of part (b) in terms of an equivalent inequality on the likelihood ratio $\Lambda(r) = p_1(r)/p_0(r)$ assumed to be a monotonically decreasing function of r.

(d) If symmetry prevails wherein $C_{00} = C_{11} = 0$ and $C_{01} = C_{10} = 1$, how do the answers in parts (b) and (c) compare with the corresponding results for the minimum average error probability decoder?