Handout 12
Wiener filter

Optimal noncausal Wiener filter

We observe $(X(t), t \in \mathbf{R})$, with $X(t) = S(t) + V(t)$, where $(S(t), t \in \mathbf{R})$ is a WSS process of interest, called the signal process, and $(V(t), t \in \mathbf{R})$ is some WSS noise. The noise and signal processes are assumed to be jointly WSS. Further, the autocorrelation functions of the signal and noise processes and the cross-correlation function between the signal and noise processes are assumed to be known. For simplicity, assume all the processes have zero mean.

Assume that we have available the entire observation $(X(u), u \in \mathbf{R})$ and, for some fixed t, it is desired to find the the LLSE (linear least mean square error) estimate $\hat{S}(t)$ of $S(t)$, i.e.

$$\hat{S}(t) = L[S(t) \mid (X(u), -\infty < u < \infty)]$$

Because of the joint wide sense stationarity of the processes involved, this estimate can be written as $\hat{S}(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du$ where $(h(u), -\infty < u < \infty)$ is some fixed function that does not depend on t, and chosen so that it minimizes $E[\mid \hat{S}(t) - S(t) \mid^2]$. Namely, $\hat{S}(t)$ is the output for input $(X(u), u \in \mathbf{R})$ of a (possibly noncausal) LTI (linear time invariant) system with an appropriately chosen impulse response $(h(u), -\infty < u < \infty)$.

By the orthogonality principle, $S(t) - \hat{S}(t)$ must be uncorrelated with $X(u)$ for all $-\infty < u < \infty$, and we must also have $E[S(t) - \hat{S}(t)] = 0$. Expanding $E[(S(t) - \hat{S}(t)) \cdot X^*(u)] = 0$ gives

$$\int_{-\infty}^{\infty} h(v)E[X(t-v)X^*(u)]dv = E[S(t)X^*(u)]$$

Setting $u = t - \tau$ results in the equation

$$\int_{-\infty}^{\infty} h(v)R_{XX}(\tau - v)dv = R_{SX}(\tau) \ , \quad \text{for all } \tau \in \mathbf{R} \ .$$

Taking the Fourier transform of this equation we get

$$H(\omega)S_{XX}(\omega) = S_{SX}(\omega) \ , \quad \text{for all } \omega \in \mathbf{R} \ .$$

This means we must have

$$H(\omega) = \frac{S_{SX}(\omega)}{S_{XX}(\omega)} \ .$$

This is the *optimal noncausal Wiener filter*.

Optimal causal Wiener filter

Consider now the problem of determining a LTI filter whose output $\hat{S}(t)$ at time t is the LLSE estimate of $S(t+\lambda)$, given the observations *up to time* t, i.e.

$$\hat{S}(t) = L[S(t+\lambda \mid (X(u), -\infty < u \leq t)]$$

If $\lambda \leq 0$ we call the problem one of *smoothing*, and if $\lambda > 0$ we call it one of *prediction*.

The problem can be rephrased as one of finding a causal impulse response $h(\cdot)$ such that $\hat{S}(t) = \int_0^\infty h(u)X(t-u)du$ minimizes $E[\mid \hat{S}(t) - S(t+\lambda) \mid^2]$. By the orthogonality principle, the error of any such estimate must be orthogonal to all the data. This gives the equations

$$E[(\hat{S}(t) - S(t+\lambda))X^*(u)] = 0 \quad \text{for all u} \leq \text{t}$$

which can be rewritten as

$$\int_0^\infty h(v)E[X(t-v)X^*(u)]dv = E[S(t+\lambda)X^*(u)] \quad \text{for all u} \leq \text{t}$$

or, by writing $u = t - \tau$, as

$$\int_0^\infty h(v)R_{XX}(\tau - v)dv = R_{SX}(\tau + \lambda) \quad \text{for all } \tau \geq 0$$

where $\tau = t - u$. Equations of this type are called Wiener-Hopf equations.

As such, this is a difficult equation to solve (our job is to solve it to find $h(\cdot)$). However, we observe that if $(X(t), t \in \mathbf{R})$ were a white noise process, its autocorrelation function would be a (scaled) δ function, and then this equation is just reading out the optimal $h(\cdot)$ as proportional to a time shift of the cross-correlation between the signal and the observations, which we assumed known. This suggests a method of attack : try to re-express the problem as one in which the observation process is white noise.

This idea cannot always be carried out. However it can be carried out if $(X(t), t \in \mathbf{R})$ has a *rational spectral density* that does not have any pure imaginary zeros. This includes most applications of practical interest. In this case we can find a causal stable LTI system with one-sided Laplace transform $L(s)$, called the *innovations filter*, having a causal stable inverse with one-sided Laplace transform $\Gamma(s)$, called the *whitening filter* such that the power spectral density of $(X(t), t \in \mathbf{R})$ is the same as that of the output of the innovations filter when its input is white noise. Since the innovations filter is causal and stable with a causal and stable inverse (such a filter is called *minimum phase*) the knowledge contained in $(X(u), u \leq t)$ for any $t \in \mathbf{R}$ is equivalent to that contained in the white noise process that is the output process of the whitening filter when its input is $(X(t), t \in \mathbf{R})$. This allows one to find the optimal filter $h(\cdot)$ by the following algorithm (when $(X(t), t \in \mathbf{R})$ has rational spectral density having no pure imaginary zeros) :

1) Factor $S_{XX}(s) = L(s)L(-s)$, where $L(s)$ is stable and causal, with stable and causal inverse $\Gamma(s) = \frac{1}{L(s)}$.

2) Let $S_{SI}(s) = \Gamma(-s)S_{SX}(s)$. This is the two sided Laplace transform of the cross-correlation of the signal with the white noise process $(I(t), t \in \mathbf{R})$ that is the output of the whitening filter when its input is $(X(t), t \in \mathbf{R})$.

3) Find $R_{SI}(\tau)$.

4) Let $g_{opt}(\tau) = R_{SI}(\tau + \lambda)u(\tau)$, where $u(\tau)$ denotes the unit step function, and let $G_{opt}(s)$ be the Laplace transform of $g_{opt}(\tau)$.

5) Let $H_{opt}(s) = \Gamma(s)G_{opt}(s)$. The optimal causal Wiener filter $h_{opt}(\tau)$ is the causal inverse Laplace transform of $H_{opt}(s)$.

An example

Suppose the received signal $X(t) = S(t) + V(t)$ where the noise $(V(t), t \in \mathbf{R})$ is WSS white noise with power spectral density $S_{VV}(s) = 3$, and the signal is a sample from a WSS process $(S(t), t \in \mathbf{R})$ with $R_{SS}(\tau) = \exp(- \mid \tau \mid)$. Further assume that the signal and noise processes are jointly WSS and $R_{SV}(\tau) = 0$.

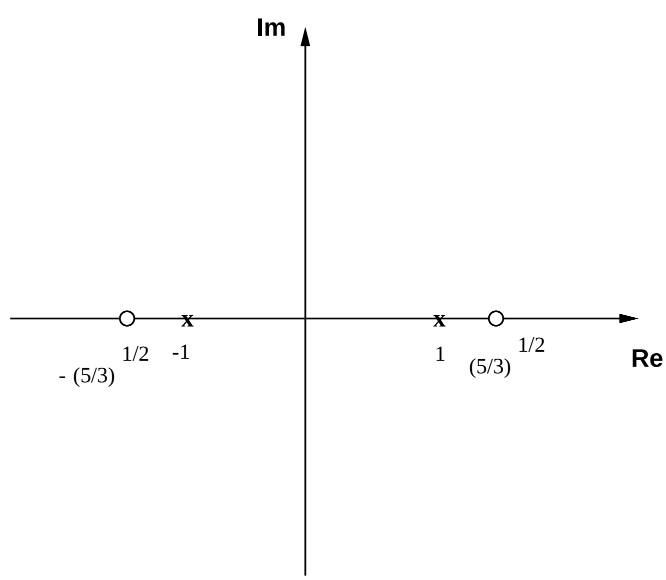
The problem is to determine the LLSE optimal causal predictor for $S(t+1)$ given $(X(u), u \leq t)$. We follow the steps above.

1)

$$S_{SS}(s) = \frac{1}{1+s} + \frac{1}{1-s} = \frac{2}{1-s^2} \ .$$

$$S_{XX}(s) = S_{SS}(s) + S_{SV}(s) + S_{VS}(s) + S_{VV}(s) = \frac{2}{1-s^2} + 0 + 0 + 3 = \frac{5-3s^2}{1-s^2} \ .$$

$S_{XX}(s)$ is a rational power spectral density, and has the following pole zero decomposition.



Picking for $L(s)$ the LHP (Left Half Plane) singularities, we see that we can write $S_{XX}(s) =$

$L(s)L(-s)$ where

$$L(s) = \frac{\sqrt{5}+\sqrt{3}s}{1+s} \quad , \qquad \Gamma(s) = \frac{1+s}{\sqrt{5}+\sqrt{3}s} \ .$$

Here $L(s)$ is stable, causal and minimum phase.

2)

$$S_{SI}(s) = \Gamma(-s)S_{SX}(s) = \Gamma(-s)S_{SS}(s) = \frac{2}{(1+s)(\sqrt{5}-\sqrt{3}s)} \ .$$

3)

$$S_{SI}(s) = \frac{a}{1+s} + \frac{b}{\sqrt{5}-\sqrt{3}s} \quad \text{where } a = \frac{2}{\sqrt{5}+\sqrt{3}} \ , b = \frac{2\sqrt{3}}{\sqrt{5}+\sqrt{3}} \ .$$

Hence, using the fact that $R_{SI}(\tau) \to 0$ as $\mid \tau \mid \to \infty$, we have

$$R_{SI}(\tau) = a\exp(-\tau)u(\tau) + \frac{1}{\sqrt{3}} b \exp(\frac{\sqrt{5}}{\sqrt{3}}\tau)u(-\tau) \ .$$

4) Here $\lambda = 1$. Hence

$$g_{opt}(\tau) = R_{SI}(\tau+1)u(\tau) = a\exp(-(\tau+1))u(\tau) \ .$$

Note that $G_{opt}(s) = \frac{a}{e(1+s)}$.

5) $H_{opt}(s) = \Gamma(s)G_{opt}(s) = \frac{a}{e(\sqrt{5}+\sqrt{3}s)}$. From this, we get

$$h_{opt}(\tau) = \frac{a}{e\sqrt{3}} \exp(-\frac{\sqrt{5}}{\sqrt{3}}\tau)u(\tau) \ .$$